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The spectrum of an I-graph

Allana S.S de Oliveira[1]
Programa de Pós-Graduação em Engenharia de Produção, UFRJ, Rio de Janeiro, RJ
Cybele T.M Vinagre[2]
Departamento de Análise, UFF, Niterói, RJ

1 Introduction

The class of I-graphs was introduced in the Foster Census [2] as a natural generalization of the so called [6] generalized Petersen graphs and has attracted the attention of many graph theorists. In our work we investigate the I-graphs under an spectral approach, which, as far as we concern, is not known.

The *adjacency matrix* $\mathbf{A}(G) = [a_{ij}]$ of an arbitrary simple graph G whose vertices are $x_1, x_2, \dots, x_n$, is the $n \times n$ matrix where $a_{ij} = 1$, if there is an edge joining x_i and x_j, and $a_{ij} = 0$ otherwise. The *characteristic polynomial* of G is that of $\mathbf{A}(G)$. An *eigenvalue of* G is any root of its characteristic polynomial. They are all real numbers. The *spectrum* of G is the set of its eigenvalues together with their multiplicities.

In our work, we completely determine the spectrum of an I-graph by using known properties of circulant and circulant block matrices.

2 Main result

Let fix $n, j, k \in \mathbb{N}$ with $n \geq 3$, $1 \leq j, k < \frac{n}{2}$ and $j \leq k$. The *I-graph* $I(n, j, k)$ is the graph with vertex set $V(I(n, j, k)) = \{a_i, b_i; 0 \leq i \leq n-1\}$ and edge set $E(I(n, j, k)) = \{\{a_i, a_{i+j}\}, \{a_i, b_i\}, \{b_i, b_{i+k}\}; 0 \leq i \leq n-1\}$, where addition is performed modulo n.

We assume $j \leq k$ since $I(n, j, k) = I(n, k, j)$. The *Petersen graph* is $I(5, 1, 2)$. The class of I-graphs contains the well known class of $G(n, k) = I(n, 1, k)$, the so called ([6]) *generalized Petersen graphs*, introduced in [3].

We denote by $A(n, j)$ the subgraph of $I(n, j, k)$ formed with the vertices $\{a_i; 0 \leq i \leq n-1\}$ and edges $\{\{a_i, a_{i+j}\}; 0 \leq i \leq n-1\}$. The subgraph of $I(n, j, k)$ with vertices $\{b_i; 0 \leq i \leq n-1\}$ and edges $\{\{b_i, b_{i+k}\}; 0 \leq i \leq n-1\}$ will be denoted $B(n, k)$. We denote $\mathbf{A}^{nj} = \mathbf{A}(A(n, j))$ and $\mathbf{B}^{nk} = \mathbf{A}(B(n, k))$.

A square matrix in which each row (after the first) has the elements of the previous row shifted cyclically one place right, is called a *circulant* matrix. We denote it as $\mathbf{M} = circ(m_0, m_1, \dots, m_{n-1})$.

[1]allanasthel@id.uff.br
[2]cybl@vm.uff.br

Lemma 2.1. $\mathbf{A}^{nj} = circ(\overbrace{0,...,0}^{j\ entries},1,0,...,0,1,\overbrace{0,...,0}^{j-1\ entries})$ *and its eigenvalues are* $\alpha_l = 2\cos(\frac{2\pi jl}{n})$, $0 \le l \le n-1$, *with corresponding eigenvectors* $\mathbf{v}_l = \left(1,\xi^l,\xi^{2l},...,\xi^{(n-1)l}\right)^T$, $0 \le l \le n-1$, *where* ξ *is a primitive* n*-root of unity.*

Analogously, $\mathbf{B}^{nk} = circ(\overbrace{0,...,0}^{k\ entries},1,0,...,0,1,\overbrace{0,...,0}^{k-1\ entries})$, *with eigenvalues* $\beta_l = 2\cos(\frac{2\pi kl}{n})$, $0 \le l \le n-1$ *and corresponding eigenvectors* $\mathbf{v}_l = (1,\xi^l,\xi^{2l},...,\xi^{(n-1)l})^T$, $0 \le l \le n-1$.

$\mathbf{A}(I(n,j,k))$ can be described as a circulant-block matrix and we establish our main result:

Theorem 2.1. *The eigenvalues of* $I(n,j,k)$ *are*

$$\lambda_l = \cos\left(\frac{2\pi jl}{n}\right) + \cos\left(\frac{2\pi kl}{n}\right) \pm \sqrt{\left(\cos\left(\frac{2\pi jl}{n}\right) - \cos\left(\frac{2\pi kl}{n}\right)\right)^2 + 1}, \quad 0 \le l \le n-1.$$

The eigenvalues of $I(n,j,k)$ *are exactly the solutions of the equations* $(\lambda-\beta_l)(\lambda-\alpha_l) = 1$, *for each* l, $0 \le l \le n-1$.

After our Theorem 2.1, we are able to prove known structural properties of I-graphs, such as conectedness and bipartiteness, by using "pure" spectral techniques.

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