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Enclosing a periodic orbit of processes in the plane

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Abstract. We give here a method for the definition of a closed continuous curve C in the $y \geq 0$ semi-plane coming from an equation of the Liénard type, in such a way that C surrounds the $y \geq 0$ - piece of a periodic orbit of the equation itself.

Keywords. Liénard equations, periodic orbit, ODE, egress points.

1 Introduction

The existence of periodic orbits for Liénard equations is one of the most considered issues in the field of the nonlinear ODE in the last fifty years. This work deals with the existence of periodic orbits in a very large class (S) of nonlinear second order differential equations of the Liénard generalized type including as special cases the classical Liénard ones, as well as the fractional power VdP equations and some TNO (truly nonlinear oscillators) considered by R. Mickens [4,5]. Through the algebraic expression concerning continuous closed curves in $\mathbb{R}^2$ by comparing slopes along such curves we make considerations on the following aspects on (S): the stability of singular points, and the existence of periodic orbits. The system (S) is represented by the equation

$$\ddot{x} - f(x)g(\dot{x}) + h(x) = 0, \tag{1}$$

where f, h are differentiable and g is continuous with $g(y) > 0$ for $y > 0$. Suppose further that f has only a finite number of zero points in which the derivative of f is non-zero and that,

$$g(0) = 0 \text{ and } h'(0) \neq 0. \tag{2}$$

The first order version of equation (1) in $\mathbb{R}^2$ is

$$\begin{cases} \dot{x} = y \\ \dot{y} = f(x)g(y) - h(x) \end{cases} . \tag{3}$$

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2 Periodic orbits

2.1 The singular point and stability

A first question we ask when designing the phase portrait in (3), concerns the stability of the critical point (0,0). Classical results using linearization in (3) are well known. One of these results says, for instance, that if $f(0)g'(0) > 2$ then the eigenvalues in the system are strictly positive and in this way, $(0,0)$ is unstable as we can see in [3]. In this direction, we can prove another result for systems in (3):

If there is $s > 0$ such that for all $0 < \epsilon < s$ we have

$$f(x)g(\sqrt{\epsilon^2 - x^2}) - h(x) + x > 0, \; x \in (-\epsilon, \epsilon),$$

then $(0,0)$ is an unstable singular point for (3). If otherwise

$$f(x)g(\sqrt{\epsilon^2 - x^2}) - h(x) + x < 0, \; x \in (-\epsilon, \epsilon),$$

$(0,0)$ is then a stable one.

In fact: the slope of the orbits in (3) through the point (x, y) are done by:

$$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx} = f(x)\frac{g(y)}{y} - \frac{h(x)}{y}.$$

Because we supposed for $y > 0$ that

$$f(x)\frac{g(y)}{y} - \frac{h(x)}{y} > -\frac{x}{y}, \tag{4}$$

and because $\dot{x} = 0$, then by naming $C_\epsilon = C_\epsilon(x, y)$, the semi-circle centered at $(0,0)$ with radius ϵ and $y > 0$, we have all orbits in (3) that cross the curve $C_\epsilon \cup (-\epsilon, 0), (\epsilon, 0)$, crossing it from the inside part [respect to the point (0,0)] to the outside one.

We showed in this way that the singular orbit $(0,0)$ is repulsive. From now on, we consider the unique singular point $(0,0)$ in the system being unstable.

2.2 The curve $\mathbf{C}_r$

If for all $(x, y) \in \mathbb{R}^2$, $\sigma^2 = x^2 + y^2$ we have along the orbits of (3):

$$\sigma\frac{\partial \sigma}{\partial x} = f(x)g(y) + x - h(x). \tag{5}$$

Let be the set $Z = a_1 < a_2 < ... < a_{n-1}$ of the zeros of f, in which f' is non-zero. Suppose that $Z \neq \emptyset$, and let be $r > 0$. Consider in this case

$$-r = a_0 < a_1 < a_2 < ... < a_{n-1} < a_n = r. \tag{6}$$

If $Z = \emptyset$ make the sequence $a_0 = -r < a_1 = r$.

Let us start defining the closed continuous curve

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$$C_r = C_r(x,y) \qquad (y \geqslant 0),$$

with $x \in [-r, a_n]$ in the clockwise sense beginning at the point $(-r, 0)$, step by step in each interval done by (6). For each point $(x,y) \in \mathbb{R}^2$, let

$$\rho^2 = x^2 + y^2,$$

and consider

$$\begin{cases} \frac{\rho_i^2(x)}{2} = S_i F(x) + \frac{x^2}{2} - H(x) + c_i & \text{in}[a_i, a_{i+1}] \ \text{ if } \ f([a_i, a_{i+1})]) \geqslant 0 \\ \frac{\rho_i^2(x)}{2} = \frac{x^2}{2} - H(x) + c_i & \text{in}[a_i, a_{i+1}] \ \text{ if } \ f([a_i, a_{i+1}]) < 0 \end{cases}, \tag{7}$$

where

$$S_i > \max\{g(y); y_i \leqslant y \leqslant y_{i+1}\} \ ; \ y_j^2 = \rho^2(a_j) - a_j^2 \ ; \ (j \in \{i, i+1\}), \tag{8}$$

and c_i is an arbitrary constant (to be determined) for $0 \leqslant i \leqslant n-1$. The functions $F(x)$, $H(x)$ are respectively the anti-derivative of $f(x)$, $h(x)$ with independent constant terms equal to zero.

Observe that the synthesis of an appropriate constant c_i in each interval $[a_i, a_{i+1}]$, $0 \leqslant i \leqslant n-1$, will be ensuring the continuity of C_r on the whole interval $[-r, a_n]$.

If an orbit γ of (3) intercept the curve C_r, $(y \geq 0)$, then by comparing (5) and (7) we have γ crossing C_r from the outside [with respect to $(0,0)$] part of the region enclosed by C_r directed to the inside one. With the aim of be showing the existence of such curve C_r we need to determine the constants r, S_i involved in the above process.

2.3 The constants r, S$_i$ (0 ≤ i ≤ n-1)

Because the curve C_r begins at $(-r, 0)$ and ends at $(a_n, 0)$ and $a_0 = -r$ we have according to (7), in a recursive mode, starting at the interval $[a_0, a_1]$ that:

$$\begin{cases} \rho_0^2(a_0) = r^2 = 2S_0F(-r) + r^2 - 2H(-r) + 2c_0 \text{ if } f > 0 \ in \ (a_0, a_1) \\ \rho_0^2(a_0) = r^2 = r^2 - 2H(-r) + 2c_0 \text{ if } f < 0 \ in \ (a_0, a_1) \end{cases},$$

and then:

$$\begin{cases} c_0 = -S_0F(-r) + H(-r) \text{ if } f > 0 \ in \ (a_0, a_1) \\ c_0 = H(-r) \text{ if } f < 0 \ in \ (a_0, a_1) \end{cases}.$$

In this way we get for $x \in [a_0, a_1]$:

$$\begin{cases} \rho_0^2(x) = 2S_0F(x) + x^2 - 2H(x) + 2(-S_0F(-r) + H(-r)) \text{ if } f > 0 \ in \ (a_0, a_1) \\ \rho_0^2(x) = -2H(x) + x^2 + 2H(-r) \text{ if } f < 0 \ in \ (a_0, a_1) \end{cases},$$

and for $x = a_1$:

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$$\begin{cases} \rho_0^2(a_1) = 2S_0(F(a_1) - F(-r)) + a_1^2 + 2(H(-r) - H(a_1)) \text{ if } f > 0 \; in \; (a_0, a_1) \\ \rho_0^2(a_1) = a_1^2 + 2(H(-r) - H(a_1)) \text{ if } f < 0 \; in \; (a_0, a_1) \end{cases}.$$

We can calculate the corresponding expressions from the next interval, $[a_1, a_2]$, till the last one, $[a_{n-1}, a_n]$, in which we must consider the restraint $\rho_{n-1}^2(a_n) = a_n^2$. Cause the curve C_r is supposed to be continuous we need to have the following conditions being satisfied:

$$r^2 = \rho_0^2(a_0), \rho_0^2(a_1) = \rho_1^2(a_1), \rho_1^2(a_2) = \rho_2^2(a_2), ..., \rho_{n-1}^2(a_n) = a_n^2. \tag{9}$$

Finally, with the purpose of having C_r well-defined, because (8) has to be accomplished, the conditions (8) and (9) actually set the link between r and S_i.

2.4 The existence of a periodic orbit

Due to the fact that $(0, 0)$ is unstable then according the Poincaré-Bendixon theorem, the existence of the curve C_k for a large enough number k shows that there exists at least on e periodic orbit of (3) enclosed by such C_k.

Further, despite some works in literature - considering particular equations in the system (S) - to be able to present an estimate for a point at which periodic orbits must cross the x-axis, providing in this way an estimate for k, we have not this possibility in general cases.

Taking on care this situation we can state the result on the existence of a periodic orbit in (3):

If for $r_0 > 0$ there exists C_r for every $r > r_0$, then (3) has at least one periodic orbit Γ.

Elsewhere by using different equations but under the same point of view that in this paper, B. C. Damasceno [2] defined a sequence of curves in $\mathbb{R}^2$ approaching indefinitely such Γ.

3 Conclusions

It was proposed in this paper a very simple method (in the sense that we only used the fundamental and almost naive analysis on crossing continuous curves in the plane) in which it was done sufficient conditions for the existence of periodic orbits for a class of Liénard equations. Notice that in [1] most profound results about the issue are done.

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