Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 5, N. 1, 2017.

Trabalho apresentado no CNMAC, Gramado - RS, 2016.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

A Locking-Free MHM Method for Elasticity

Weslley S. Pereira[1]
Frédéric Valentin[2]
Coordenação de Matemática Aplicada e Computacional, LNCC, Petrópolis, RJ

Abstract. This work presents a multiscale hybrid-mixed finite element (MHM) method for the two- and three-dimensional linear elasticity problem that deals with nearly incompressible and heterogeneous isotropic materials. The starting point is a dual-hybrid form of the elasticity model defined on a coarse mesh, which is equivalent to a set of element-wise elasticity problems brought together by a face-based global formulation. Importantly, the local problems are independent to one another and determine the basis functions. Thereby, the basis naturally incorporate multiscale features of the media. This new variant of the MHM method turns out to be robust in the incompressible limit case as a result of the use of a stabilized finite element method to approximate basis functions. Some preliminary theoretical results are addressed.

Keywords. Elasticity, Incompressibility, Hybridization, Stabilized method, Multiscale

1 Introduction

In the last decade, there has been a great development of massively parallel systems characterized by computers with a large numbers of processors (grouped in cores) of moderate speed and storage capacities. These new technologies have changed what practitioners expect from numerical methods in terms of performance. Indeed, although precision and robustness remain fundamental properties of numerical methods, for extreme-scale computational science, the underlying algorithms must take fully advantage of the new massive parallel architectures. In this context, the concept of "divide and conquer" emerges as the natural candidate to drive the development of new numerical methods. Multiscale finite element methods embed such philosophy in their construction. In a broad sense, multiscale methods are built to be precise on coarse meshes by upscaling missing unresolved scales structures of the solution into the basis functions. These so-called multiscale basis functions are computed from independent local problems defined element-wisely.

The Multiscale Hybrid-Mixed (MHM) method is a member of the family of multiscale finite element methods. It was first introduced for the transport equation in [1, 5, 6, 7]. The MHM method has been also applied to the elasticity problem in [4]. In [4] the starting point is the primal-hybrid formulation of the elasticity problem, i.e., the continuity of the displacement is relaxed on element boundaries using Lagrange multipliers. As such,

[1]weslleyp@lncc.br
[2]valentin@lncc.br

the numerical solution is obtained solving a set of independent, element-wise elasticity problems which are coupled with a global problem defined on the skeleton of the partition. Since the basis functions are computed from the second-order elasticity model at the local level, the standard Galerkin method based on piecewise polynomial spaces was used to approximate the local solutions in [4].

It is well known that standard low order finite element schemes suffer from the "locking" phenomena when they are applied to nearly incompressible material problems (Poisson ratio close to 1/2) [2]. One way to circumvent this difficulty is to rewrite the elasticity model in its mixed counterpart making resorting the stress tensor as an independent variable. Another possibility is to introduce an extra "pressure" variable, mimicking what is done in fluids. In both cases, the quasi-incompressible aspect of the model makes the choice of pair of approximation spaces non-trivial. Particularly, the (appealing) equal order polynomial spaces may not be adopted. A classical approach to overcome such a limitation consists of employing a stabilized finite element method (see [3]).

This work extends the MHM method proposed in [4] to the nearly incompressible isotropic elasticity model. To this end, we adopt the elasticity problem written with respect to the displacement and pressure variables, and propose a variant of the Galerkin Least-Squares (GLS) stabilized method [3] to approximate the multiscale basis functions. The development of the new MHM method is presented in a constructive way, and we anticipate some theoretical results as the well-posedness and the local conservation properties of the method.

2 The linear isotropic elasticity model

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be an open bounded domain with polygonal boundary $\partial\Omega$. The linear elasticity problem consists of finding the displacement $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ such that

$$\begin{cases} -\mathbf{div}\,(\mathsf{C}\,\mathcal{E}(\boldsymbol{u})) = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where C denotes the fourth-order stiffness tensor which acts on the space $\mathbb{R}^{d\times d}_{sym}$ of $d \times d$ positive definite symmetric matrices with values in $\mathbb{R}^{d\times d}_{sym}$, $\mathcal{E}(\boldsymbol{u})$ denotes the infinitesimal strain tensor, i.e., the symmetric part of the deformation gradient of $\boldsymbol{u}$, $\boldsymbol{f}$ is the distributed load due to body forces, and $\sigma := \mathsf{C}\,\mathcal{E}(\boldsymbol{u})$ represents the stress tensor. Hereafter, we assume $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\partial\Omega)$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ such that problem (1) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, and $\sigma \in H(\mathbf{div};\Omega)$ (the spaces having their usual meanings.)

The stiffness tensor is quite general, possibly embedding multiple geometrical scales on Ω, which is assumed to be uniformly elliptic and bounded. Here we are interested in the application of the MHM method to isotropic elastic media. As a result, the stiffness tensor C can be characterized by the shear modulus μ and the Poisson's ratio ν as follows

$$\mathsf{C}_{ijkl} := 2\,\mu\,\delta_{ik}\,\delta_{jl} + \frac{2\,\mu\,\nu}{1-2\,\nu}\delta_{ij}\,\delta_{kl} \quad \text{or} \quad \mathsf{C}\,\mathcal{E}(\boldsymbol{u}) = 2\,\mu\,\mathcal{E}(\boldsymbol{u}) + \frac{2\,\mu\,\nu}{1-2\,\nu}(\operatorname{div}\boldsymbol{u})I, \tag{2}$$

DOI: 10.5540/03.2017.005.01.0336

where δ_{ij} denotes the Kronecker's delta and I is the identity tensor. Here we suppose $\mu \in W^{1,\infty}(\Omega)$ and $\nu \in L^{\infty}(\Omega)$. To simplify further notation we define

$$\varepsilon := \frac{1-2\nu}{2\mu\nu}. \tag{3}$$

We assume $\mu \geqslant \mu_0$, $\nu \in [\nu_0, 1/2)$ and $\epsilon \geqslant \epsilon_0$, where μ_0, ν_0 and ϵ_0 are positive constants.

We recall that a linearly isotropic material is said to be nearly incompressible if ν approaches $1/2$ (e.g. $\varepsilon \to 0$). It is well-known that the most common (and used) low order finite element methods show poor performance in such a case, a phenomena called *locking* (c.f. [2]). A way to overcome such a numerical drawback consists of introducing the scalar "pressure" field and rewriting the second-order isotropic elasticity model (1) in the following equivalent form: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ such that

$$\begin{cases} -\mathbf{div}\,(2\,\mu\,\mathcal{E}(\boldsymbol{u}) - p\,I) \;=\; \boldsymbol{f} & \text{in } \Omega\,, \\ \varepsilon\,p + \operatorname{div}\boldsymbol{u} \;=\; 0 & \text{in } \Omega\,, \\ \boldsymbol{u} \;=\; \boldsymbol{g} & \text{on } \partial\Omega\,, \end{cases} \tag{4}$$

and, then, propose a numerical method to approximate the solution of (4) in the incompressible limit case ($\varepsilon \to 0$) [3].

In [4], a MHM method was proposed to problem (1) using usual finite element methods to approximate the basis functions at the element level. In this work, we shall revisit the MHM methodology and propose a *new* robust MHM method for vanishing ε. This is addressed in the next sections.

2.1 The hybrid formulation

Instead of working directly with the standard weak formulation of problem (4), we seek the displacement $\boldsymbol{u}$ in a weaker space which relaxes its continuity. Specifically, let $\{\mathcal{T}_H\}_{H>0}$ be a family of regular meshes that partition Ω, where H is the characteristic length of $\mathcal{T}_H$. Without loss of generality, we use here the terminology usually employed for three-dimensional domains. The boundary ∂K of an element $K \in \mathcal{T}_H$ is formed by faces F and $\boldsymbol{n}^K$ denotes the outward unitary vector on ∂K. The space of displacements is given as follows

$$\mathbf{V} := \boldsymbol{H}^1(\mathcal{T}_H) := \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \,:\, \boldsymbol{v}|_K \in \boldsymbol{H}^1(K), \quad \forall K \in \mathcal{T}_H \right\}\,, \tag{5}$$

the space of tractions $\boldsymbol{\Lambda}$ is formed as follows

$$\boldsymbol{\Lambda} := \left\{ \sigma\,\boldsymbol{n}^K|_{\partial K} \in \boldsymbol{H}^{-1/2}(\partial K), \quad \forall K \in \mathcal{T}_H \,:\, \sigma \in H(\mathbf{div};\Omega) \right\}. \tag{6}$$

The space of pressures is $Q := L^2(\Omega)$. If D is an open set, we denote the inner product in $L^2(D)$ (e.g. $\boldsymbol{L}^2(D)$ and $[L^2(D)]^{d\times d}$) by $(\cdot,\cdot)_D$ and the $\boldsymbol{H}^{-1/2}(\partial D) \times \boldsymbol{H}^{1/2}(\partial D)$ duality product by $\langle \cdot,\cdot\rangle_{\partial D}$. Also, we define the following products

$$(\cdot,\cdot)_{\mathcal{T}_H} := \sum_{K\in\mathcal{T}_H} (\cdot,\cdot)_K \quad \text{and} \quad \langle\cdot,\cdot\rangle_{\partial\mathcal{T}_H} := \sum_{K\in\mathcal{T}_H} \langle\cdot,\cdot\rangle_{\partial K}\,. \tag{7}$$

DOI: 10.5540/03.2017.005.01.0336

The hybrid formulation of problem (4) reads: Find $(\boldsymbol{u},\, p,\, \boldsymbol{\lambda}) \in \mathbf{V} \times Q \times \boldsymbol{\Lambda}$ such that

$$\begin{cases} (2\,\mu\,\mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{v}))_{\mathcal{T}_H} - (p, \operatorname{div}\,\boldsymbol{v})_{\mathcal{T}_H} + \langle \boldsymbol{\lambda}, \boldsymbol{v}\rangle_{\partial\mathcal{T}_H} = (\boldsymbol{f},\, \boldsymbol{v})_{\mathcal{T}_H} & \text{for all } \boldsymbol{v} \in \mathbf{V}\,, \\ (\varepsilon\, p, q)_{\mathcal{T}_H} + (\operatorname{div}\boldsymbol{u}, q)_{\mathcal{T}_H} = 0 & \text{for all } q \in Q\,, \\ \langle \boldsymbol{\mu}, \boldsymbol{u}\rangle_{\partial\mathcal{T}_H} = \langle \boldsymbol{\mu},\, \boldsymbol{g}\rangle_{\partial\Omega} & \text{for all } \boldsymbol{\mu} \in \boldsymbol{\Lambda}\,. \end{cases} \tag{8}$$

The next result addresses the equivalence between the weak form of (4) and (8).

Lemma 2.1. *Assume that* $(\boldsymbol{u}, p, \boldsymbol{\lambda}) \in \mathbf{V} \times Q \times \boldsymbol{\Lambda}$. *Therefore,* $(\boldsymbol{u}, p, \boldsymbol{\lambda})$ *is the solution of* (8) *if and only if* $(\boldsymbol{u}, p) \in \boldsymbol{H}^1(\Omega) \times Q$ *solves* (1) *(in a distributional sense). Furthermore,* $\boldsymbol{\lambda}$ *is characterized in* $K \in \mathcal{T}_H$ *as follows*

$$\boldsymbol{\lambda} \,=\, -(2\,\mu\,\mathcal{E}(\boldsymbol{u}) - p\,I)\,\boldsymbol{n}^K \quad \text{on } \partial K\,. \tag{9}$$

Proof. Problem (8) is a particular case of the general hybrid formulation proposed in [4]. Therefore, this result follows, as in [4], from the proof of Theorem 1 in [8]. □

Before going discrete, we propose an equivalent form to problem (8) which is more suitable to develop "divide and conquer" algorithms.

2.2 An equivalent global-local formulation

First, observe that the space $\mathbf{V}$ in (5) can be decomposed as

$$\mathbf{V} = \mathbf{V}_{rm} \oplus \widetilde{\mathbf{V}}\,, \tag{10}$$

where $\mathbf{V}_{rm} := \oplus_{K\in\mathcal{T}_H} \mathbf{V}^K_{\mathrm{rm}}$ is the space of piecewise rigid body modes over K, i.e., $\mathcal{E}(\boldsymbol{v}^{\mathrm{rm}})\,|_K = 0$ for all $\boldsymbol{v}^{\mathrm{rm}} \in \mathbf{V}^K_{\mathrm{rm}}$, and $\widetilde{\mathbf{V}}$ is its L^2-orthogonal complement. Hereafter, X^K denotes the restriction of a space X to an element $K \in \mathcal{T}_H$.

Following closely [4], we define $T = (T^u, T^p) : \boldsymbol{\Lambda} \to \widetilde{\mathbf{V}} \times Q$ and $\hat{T} = (\hat{T}^u, \hat{T}^p) : \boldsymbol{L}^2(\Omega) \to \widetilde{\mathbf{V}} \times Q$ as linear bounded operators, such that, for all $K \in \mathcal{T}_H$, $T\boldsymbol{\mu}\,|_K$ satisfies

$$\begin{cases} \big(2\,\mu\,\mathcal{E}(T^u\,\boldsymbol{\mu}), \mathcal{E}(\tilde{\boldsymbol{v}})\big)_K - (T^p\,\boldsymbol{\mu}, \operatorname{div}\,\tilde{\boldsymbol{v}})_K = -\langle \boldsymbol{\mu}, \tilde{\boldsymbol{v}}\rangle_{\partial K} & \text{for all } \tilde{\boldsymbol{v}} \in \widetilde{\mathbf{V}}^K\,, \\ (\varepsilon\, T^p\,\boldsymbol{\mu}, q)_K + (\operatorname{div}(T^u\,\boldsymbol{\mu}), q)_K = 0 & \text{for all } q \in Q^K\,, \end{cases} \tag{11}$$

and $\hat{T}\boldsymbol{\mu}\,|_K$ satisfies

$$\begin{cases} \big(2\,\mu\,\mathcal{E}(\hat{T}^u\,\boldsymbol{f}), \mathcal{E}(\tilde{\boldsymbol{v}})\big)_K - (\hat{T}^p\,\boldsymbol{f}, \operatorname{div}\,\tilde{\boldsymbol{v}})_K = (\boldsymbol{f}, \tilde{\boldsymbol{v}})_K & \text{for all } \tilde{\boldsymbol{v}} \in \widetilde{\mathbf{V}}^K\,, \\ (\varepsilon\, \hat{T}^p\,\boldsymbol{f}, q)_K + (\operatorname{div}(\hat{T}^u\,\boldsymbol{f}), q)_K = 0 & \text{for all } q \in Q^K\,. \end{cases} \tag{12}$$

Next, from (11)-(12), we can rewrite (8) as follows: Find $(\boldsymbol{u}^{\mathrm{rm}}, \boldsymbol{\lambda}) \in \mathbf{V}_{\mathrm{rm}} \times \boldsymbol{\Lambda}$ such that

$$\begin{cases} \langle \boldsymbol{\lambda}, \boldsymbol{v}^{\mathrm{rm}}\rangle_{\partial\mathcal{T}_H} = (\boldsymbol{f}, \boldsymbol{v}^{\mathrm{rm}})_{\mathcal{T}_H} & \text{for all } \boldsymbol{v}^{\mathrm{rm}} \in \mathbf{V}_{\mathrm{rm}}\,, \\ \langle \boldsymbol{\mu}, \boldsymbol{u}^{\mathrm{rm}} + T^u\boldsymbol{\lambda}\rangle_{\partial\mathcal{T}_H} = -\langle \boldsymbol{\mu}, \hat{T}^u\boldsymbol{f}\rangle_{\partial\mathcal{T}_H} + \langle \boldsymbol{\mu},\, \boldsymbol{g}\rangle_{\partial\Omega} & \text{for all } \boldsymbol{\mu} \in \boldsymbol{\Lambda}\,. \end{cases} \tag{13}$$

The coupled local-global system (11)-(13) is equivalent to the hybrid formulation (8). This is addressed in the next lemma.

Lemma 2.2. *Function* $(\boldsymbol{u}^{rm}, \boldsymbol{\lambda}) \in \mathbf{V}_{rm} \times \boldsymbol{\Lambda}$ *is the unique solution of problem* (13) *if and only if function* $(\boldsymbol{u}, p, \boldsymbol{\lambda})$ *is the unique solution of problem* (8)*. Furthermore, the following characterization holds*

$$\boldsymbol{u} = \boldsymbol{u}^{rm} + T^u\,\boldsymbol{\lambda} + \hat{T}^u\,\boldsymbol{f} \quad \textit{and} \quad p = T^p\,\boldsymbol{\lambda} + \hat{T}^p\,\boldsymbol{f}\,. \tag{14}$$

Proof. Notice that (8) and (13) fit the abstract forms (7) and (11) in [7], respectively. Therefore, this result follows from Theorem 1 in [7]. □

Next, we introduce the MHM method which corresponds to discretize (11)-(13).

3 The MHM method

Let $\boldsymbol{\Lambda}_H \subset \boldsymbol{\Lambda}$ be the finite element space defined by

$$\boldsymbol{\Lambda}_h = \boldsymbol{\Lambda}_l := \left\{ \boldsymbol{\mu} \in \boldsymbol{\Lambda} \,:\, \boldsymbol{\mu}\,|_F \in [\mathbb{P}^l(F)]^d, \quad \forall F \in \partial\mathcal{T}_H \right\}, \tag{15}$$

where $\mathbb{P}^l(F)$ is the space of piecewise polynomial functions of degree equal or less than $l \geq 1$ on F. Assuming that the action of T^u on the (finite dimensional) basis of $\boldsymbol{\Lambda}_H$ and $\hat{T}^u\,\boldsymbol{f}$ are known exactly, we obtain the so-called one-level MHM method by replacing $\boldsymbol{\Lambda}$ by $\boldsymbol{\Lambda}_H$ in (13). Generally, closed formulas are not available for the solution of the local problems, although some cases exist (observe that $\hat{T}^u\,\boldsymbol{f} = 0$ if $\boldsymbol{f} \in \mathbf{V}_{rm}$, for instance). Therefore, a second level of discretization is mandatory, which corresponds to devise an accurate approximations for operators T and $\hat{T}$. This results in a two-level MHM method.

In [4], the classical Galerkin method defined over piecewise polynomial spaces was adopted to approximate T^u and $\hat{T}^u$. Such operators are defined through local elasticity problems written in terms of the displacement variable only. Thereby, no strategy to overcome the “locking” issue was addressed. Unlike [4], the local problems (11)-(12) are now given in terms of the displacement-pressure variables. Such a feature allows us to propose a locking-free GLS method within the MHM methodology. This GLS method is a variant of the one presented in [3].

First, we need some extra notations. Given $K \in \mathcal{T}_H$, let $\mathcal{T}_h^K$ be a regular partition of K composed of elements $\tau \in \mathcal{T}_h^K$. We define the finite dimensional subspaces $\widetilde{\mathbf{V}}_h^K \subset \widetilde{\mathbf{V}}^K$ and $Q_h^K \subset Q^K$ by

$$\widetilde{\mathbf{V}}_h^K := \left\{ \tilde{\boldsymbol{v}}_h \in \widetilde{\mathbf{V}}^K \cap C^0(K) \,:\, \tilde{\boldsymbol{v}}_h\,|_\tau \in [\mathbb{S}^k(\tau)]^d, \quad \forall \tau \in \mathcal{T}_h^K \right\}, \tag{16}$$

$$Q_h^K := \left\{ q_h \in C^0(K) \,:\, q_h\,|_\tau \in \mathbb{S}^k(\tau), \quad \forall \tau \in \mathcal{T}_h^K \right\}, \tag{17}$$

with a constant $k \geqslant 1$, where $\mathbb{S}^k(\tau) := \mathbb{P}^k(\tau)$ or $\mathbb{Q}^k(\tau)$, and $\mathbb{P}^k(\tau)$ is the space of polynomial functions of degree less or equal to k and $\mathbb{Q}^k(\tau)$ is the space of tensor polynomial functions of order k at most. The global discrete spaces are $\widetilde{\mathbf{V}}_h := \oplus_{K \in \mathcal{T}_H} \widetilde{\mathbf{V}}_h^K$ and $Q_h = \oplus_{K \in \mathcal{T}_H} Q_h^K$.

In this work, we propose the following two-level MHM method for problem (11)-(13): Find $(\boldsymbol{u}_H^{rm}, \boldsymbol{\lambda}_H) \in \mathbf{V}_{rm} \times \boldsymbol{\Lambda}_H$ such that

$$\left\{ \begin{array}{c} \langle \boldsymbol{\lambda}_H, \boldsymbol{v}^{rm} \rangle_{\partial\mathcal{T}_H} = (\boldsymbol{f}, \boldsymbol{v}^{rm})_{\mathcal{T}_H} \quad \text{for all } \boldsymbol{v}^{rm} \in \mathbf{V}_{rm}\,, \\ \langle \boldsymbol{\mu}_H, \boldsymbol{u}_H^{rm} + T_h^u\,\boldsymbol{\lambda}_H \rangle_{\partial\mathcal{T}_H} = -\langle \boldsymbol{\mu}_H, \hat{T}_h^u\,\boldsymbol{f} \rangle_{\partial\mathcal{T}_H} + \langle \boldsymbol{\mu}_H, \boldsymbol{g} \rangle_{\partial\Omega} \text{ for all } \boldsymbol{\mu}_H \in \boldsymbol{\Lambda}_H\,. \end{array} \right. \tag{18}$$

The linear operators $T_h = (T_h^u, T_h^p) : \mathbf{\Lambda} \to \widetilde{\mathbf{V}}_h \times Q_h$ and $\hat{T}_h = (\hat{T}_h^u, \hat{T}_h^p) : \boldsymbol{L}^2(\Omega) \to \widetilde{\mathbf{V}}_h \times Q_h$ approximate T and $\hat{T}$, and are defined in each $K \in \mathcal{T}_H$ through

$$B_K(T_h^u\,\boldsymbol{\mu}, T_h^p\,\boldsymbol{\mu}; \tilde{\boldsymbol{v}}_h, q_h) = F_K^{\boldsymbol{\mu}}(\tilde{\boldsymbol{v}}_h, q_h) \qquad \text{for all } (\tilde{\boldsymbol{v}}_h, q_h) \in \widetilde{\mathbf{V}}_h^K \times Q_h^K, \tag{19}$$

$$B_K(\hat{T}_h^u\,\boldsymbol{q}, \hat{T}_h^p\,\boldsymbol{q}; \tilde{\boldsymbol{v}}_h, q_h) = F_K^{\boldsymbol{q}}(\tilde{\boldsymbol{v}}_h, q_h) \qquad \text{for all } (\tilde{\boldsymbol{v}}_h, q_h) \in \widetilde{\mathbf{V}}_h^K \times Q_h^K, \tag{20}$$

where $B_K : [\widetilde{\mathbf{V}}_h^K \times Q_h^K] \times [\widetilde{\mathbf{V}}_h^K \times Q_h^K] \to \mathbb{R}$ and $F_K^{\boldsymbol{\mu}}$, $F_K^{\boldsymbol{q}} : \widetilde{\mathbf{V}}_h^K \times Q_h^K \to \mathbb{R}$ are given by

$$\begin{aligned} B_K(\tilde{\boldsymbol{u}}, p; \tilde{\boldsymbol{v}}, q) &= (2\,\mu\,\mathcal{E}(\tilde{\boldsymbol{u}}), \mathcal{E}(\tilde{\boldsymbol{v}}))_K - (p, \operatorname{div}\,\tilde{\boldsymbol{v}})_K - (\varepsilon\, p, q)_K - (\operatorname{div}\,\tilde{\boldsymbol{u}}, q)_K \\ &\quad - \alpha \sum_{\tau \in \mathcal{T}_h^K} \frac{h_\tau^2}{\|2\,\mu\|_{\infty,\tau}} (-\mathbf{div}\,(2\,\mu\,\mathcal{E}(\tilde{\boldsymbol{u}}) - pI), -\mathbf{div}\,(2\,\mu\,\mathcal{E}(\tilde{\boldsymbol{v}}) - qI))_\tau\,, \end{aligned} \tag{21}$$

$$F_K^{\boldsymbol{\mu}}(\tilde{\boldsymbol{v}}, q) = -\langle \boldsymbol{\mu}, \tilde{\boldsymbol{v}} \rangle_{\partial K}\,, \tag{22}$$

$$F_K^{\boldsymbol{q}}(\tilde{\boldsymbol{v}}, q) = (\boldsymbol{q}, \tilde{\boldsymbol{v}})_K - \alpha \sum_{\tau \in \mathcal{T}_h^K} \frac{h_\tau^2}{\|2\,\mu\|_{\infty,\tau}} (\boldsymbol{q}, -\mathbf{div}\,(2\,\mu\,\mathcal{E}(\tilde{\boldsymbol{v}}) - qI))_\tau\,. \tag{23}$$

Here α is positive constant, h_τ denotes the diameter of $\tau \in \mathcal{T}_h^K$, and $\|\cdot\|_{\infty,\tau}$ is the $L^\infty(\tau)$-norm. Now, owing to solutions $\boldsymbol{u}_H^{\mathrm{rm}}$ and $\boldsymbol{\lambda}_H$ from (18), we build an approximation of the exact solutions $\boldsymbol{u}$, p and σ through the following discrete functions

$$\boldsymbol{u}_h = \boldsymbol{u}_H^{\mathrm{rm}} + T_h^u\,\boldsymbol{\lambda}_H + \hat{T}_h^u\,\boldsymbol{f}\,, \quad p_h = T_h^p\,\boldsymbol{\lambda}_H + \hat{T}_h^p\,\boldsymbol{f}\,, \quad \sigma_h = 2\,\mu\,\mathcal{E}(\boldsymbol{u}_h) - p_h\,I\,. \tag{24}$$

Next, we address a sufficient condition for well-posedness of the two-level method (18).

Theorem 3.1. *Let $\mathcal{N}_H := \{\boldsymbol{\mu}_H \in \mathbf{\Lambda}_l : \langle \boldsymbol{\mu}_H, \boldsymbol{v}^{\mathrm{rm}} \rangle_{\partial\mathcal{T}_H} = 0\,,\ \forall\,\boldsymbol{v}^{\mathrm{rm}} \in \mathbf{V}_{\mathrm{rm}}\}$. Assume α is small enough (not depending on ϵ) and the following compatibility condition holds:*

$$\forall \boldsymbol{\mu}_H \in \mathcal{N}_H, \quad \langle \boldsymbol{\mu}_H, \tilde{\boldsymbol{v}}_h \rangle_{\partial K} = 0 \quad \textit{for all } \tilde{\boldsymbol{v}}_h \in \widetilde{\mathbf{V}}_h \textit{ and } K \in \mathcal{T}_H \quad \Rightarrow \quad \boldsymbol{\mu}_H = 0\,. \tag{25}$$

Then, the MHM method (18) *is well-posed.*

Proof. Using an adaption of Lemma 3.2 from [3], we conclude T_h and $\hat{T}_h$ are bounded if α is small enough. Therefore the result follows closely to the proof of Theorem 6.2 in [4]. □

Remark 3.1. *It holds from* (18) *that the approximate traction $\boldsymbol{\lambda}_H$ is in local equilibrium with respect to external forces, i.e., it satisfies the following problem*

$$\int_{\partial K} \boldsymbol{\lambda}_H \cdot \boldsymbol{v}^{\mathrm{rm}}\, d\boldsymbol{x} = \int_K \boldsymbol{f} \cdot \boldsymbol{v}^{\mathrm{rm}}\, d\boldsymbol{x} \quad \textit{for all } \boldsymbol{v}^{\mathrm{rm}} \in \mathbf{V}_{\mathrm{rm}}\,. \tag{26}$$

Also, observe that testing $\tilde{\boldsymbol{v}}_h = \mathbf{0}$ and $q_h = 1$ in (19) *and* (20)*, respectively, the following local compressibility constraint is fulfilled*

$$\int_K \operatorname{div}\,\boldsymbol{u}_h + \varepsilon\, p_h\, d\boldsymbol{x} = 0\,. \tag{27}$$

Remark 3.2. *From the algorithmic viewpoint, local problems* (19)*-*(20) *can be decoupled from the global one* (18) *as follows: we first solve problems* (19)*, for each $K \in \mathcal{T}_H$ and each basis function $\boldsymbol{\psi}$ in $\mathbf{\Lambda}_H$, and* (20) *with $\boldsymbol{q} = \boldsymbol{f}$. Then we solve the global problem* (18) *to get the degrees of freedom of $\boldsymbol{\lambda}_h$ and $\boldsymbol{u}_H^{\mathrm{rm}}$ using the computed multiscale basis function $T_h^u\boldsymbol{\psi}$ and $\hat{T}_h^u\boldsymbol{f}$. Collecting these results, we obtain the numerical solutions from* (24).

DOI: 10.5540/03.2017.005.01.0336

4 Conclusions

A new MHM method for the two- and three-dimensional linear isotropic elasticity was proposed in this work. It was particularly designed to handle heterogeneous quasi-incompressible materials, a property inherited from the stabilized finite element method used at the second level to approximate the multiscale basis functions. It is worthy mentioning that the underlying algorithm associated to the MHM method is naturally adapted to the new generation of massive parallel computers since the local problems can be solved independently of one another. The numerical analysis of the method as well as its numerical validation were left out of the scope of this work. These subjects will be addressed in a forthcoming work.

Acknowledgements

This work was funded by CNPq/Brazil (301576/2013-0, 480634/2013-0, 140764/2015-1), FAPERJ/Brazil, RNP/Brazil and BULL/France.

References

[1] R. Araya, C. Harder, D. Paredes, and F. Valentin. Multiscale hybrid-mixed method. *SIAM J. Numer. Anal.*, 513505–3531, 2013. DOI: 10.1137/120888223.

[2] I. Babuška and M. Suri. Locking effects in the finite element approximation of elasticity problems. *Numer. Math.*, 62:439–463, 1991. DOI: 10.1007/BF01396238.

[3] L. P. Franca and R. Stenberg. Error analysis of some galerkin least squares methods for the elasticity equations. *SIAM J. Numer. Anal.*, 28:1680–1697, 1991. ISSN: 0036-1429.

[4] C. Harder, A. L. Madureira, and F. Valentin. A hybrid-mixed method for elasticity. *ESAIM: M2AN*, 50:311–336, 2016. DOI: 10.1051/m2an/2015046.

[5] C. Harder, D. Paredes, and F. Valentin. A family of multiscale hybrid-mixed finite element methods for the darcy equation with rough coefficients. *J. Comput. Phys.*, 245:107–130, 2013. DOI: 10.1016/j.jcp.2013.03.019.

[6] C. Harder, D. Paredes, and F. Valentin. On a multiscale hybrid-mixed method for advective-reactive dominated problems with heterogeneous coefficients. *Multiscale Model. Simul.*, 13:491–518, 2015. DOI: 10.1137/130938499.

[7] C. Harder and F. Valentin. Foundations of the MHM method. *Building Bridges: Connections and Challenges in Modern Approaches to Numerical Partial Differential Equations*, Lecture Notes in Computational Science and Engineering. Springer, 2016. (to appear)

[8] P. Raviart and J. Thomas. Primal hybrid finite element methods for 2nd order elliptic equations. *Math. Comp.*, 31:391–413, 1977. DOI: 10.2307/2006423.

