Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 5, N. 1, 2017.

Trabalho apresentado no CNMAC, Gramado - RS, 2016.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

On a high-order numerical scheme for the Lippmann-Schwinger equation in layered medium

Wagner B. Muniz [1]

Department of Mathematics, Federal University of Santa Catarina (UFSC), Brazil

Abstract. We consider the direct wave scattering problem from a penetrable obstacle located either in a homogeneous or in a layered background, motivated for example by the simulation of propagation from ultrasound or from buried electromagnetic material via GPR (ground-penetrating radar) or electromagnetic induction devices. Starting from a volume integral equation (the Lippmann-Schwinger equation), we devise a collocation method in which the singularity is analytically treated, and the basis functions for the remainder are piecewise continuous polynomials of arbitrary degree. This allows the simulation of scattered fields due to penetrable obstacles with spatially varying permittivity and conductivity, for which some examples are discussed.

Keywords. Lippmann-Schwinger equation, wave scattering, collocation method, Duffy transformation

1 Introduction

Let $G(x,y)$ be the two-dimensional Green's function of the layered medium and k_-, k_+ be the wave numbers of the bottom and top layers respectively (e.g. soil and air), where $\Im m\, k \geq 0$. If we consider the TM polarization ($u = E_z$) then the total field satisfies the well-known Lippmann-Schwinger equation

$$(I + k_-^2 T_m)u(x) \quad = \quad u^i(x), \qquad x \in \mathbb{R}^2, \tag{1}$$

where $m(x) = 1 - n(x)$ and $n(x)$ is the refractive index characterizing the non-homogeneity of an obstacle D (the pertubation from the background), the incident field, denoted by $u^i = u^{inc}$, is either a plane wave or a point source located in the upper layer

$$u^i(x) := G(x,z), \quad z \in \mathbb{R}^2_+$$

and, for an integrable density ϕ in D,

$$(T_m\phi)(x) := \int_D m(y)\phi(y)G(x,y)dy, \quad m := 1 - n.$$

From the Lippmann-Schwinger equation [1, 2], we see that once the internal total field $u(x), x \in D$, is known, the scattered field outside the scatterer is given by

$$u^s(x) \quad = \quad -k_-^2(T_m u)(x).$$

[1]w.b.muniz@ufsc.br

Scaling through the refractive index

Let $D := \{x \in \mathbb{R}^2 | m(x) \neq 0\}$ and u be a solution of (1), then multiplying (1) by $m(x)$ we have

$$m(x)u(x) + k_-^2\, m(x) \int_D m(y)u(y)G(x,y)dy = m(x)u^i(x), \quad x \in D.$$

Therefore, letting

$$v(x) := m(x)u(x), \text{ and } f(x) := m(x)u^i(x), \qquad x \in D$$

we see that v solves

$$(I + k_-^2 m(x)T)v(x) \;\; = \;\; f(x), \quad x \in D,$$

where $T\phi = T_1\phi$. In free-space, one can simplify (1) further by re-scaling $x' = kx$ so that the volume integral equation involves only the 2D free-space fundamental solution $\mathrm{i}/4H_0^{(1)}(|x' - y'|)$ and the integral domain is $D' = kD$, where $H_0^{(1)}(\cdot)$ is the zeroth order Hankel function of the first kind. A similar procedure could be applied here, but the space integrals would have to be evaluated in the complex plane with an appropriate transformation.

2 A collocation method

Let $\tau_h = \{K_j : j = 1, \ldots, N\}$ be a triangulation of D and

$$v(x) \;\; = \;\; \sum_{j=1}^{N} a_j \chi_j(x), \tag{2}$$

where χ_j is a basis function supported in K_j. For sake of simplicity we assume that X_j is the indicator function of K_j. By *collocating* at the centers x_i of the triangles K_i and letting $B_{ij} := \int_{K_j} G(x_i, y)dy$ be the integral over the triangle K_j, the Lippmann-Schwinger equation

$$v(x) + k_-^2 m(x) \int_D G(x,y)v(y)dy \;\; = \;\; f(x), \quad x \in D,$$

reduces to the discrete system

$$a_i + k_-^2 m_i \sum_j B_{ij} a_j \;\; = \;\; f_i,$$

where $f_i := f(x_i), a_i := v(x_i)$ and the refractive index is assumed constant inside each element. Letting $A_{ij} := k_-^2 m_i B_{ij}$, we have

$$(\boldsymbol{I} + \boldsymbol{A})\vec{a} = \vec{f}.$$

We now turn out attention to the computation of B_{ij}. The main concern here is with the singularity of the integrand over triangular elements, the non-singular part of the integrand can be handled numerically in a standard manner. What we show next is how to integrate the singularity analytically over an *arbitrary triangle.* We start by recalling the asymptotic behavior of the Green's function as follows

Remark 2.1. *We recall that*

$$G(x,y) := \frac{\mathrm{i}}{4}H_0^{(1)}(k_-|x-y|) + G_0(x,y),$$

where $G_0(x,y)$ *is smooth, and as* $|x-y| \to 0$, G *has a logarithmic singularity driven by the Hankel function. In fact, as* $|x-y| \to 0$,

$$G(x,y) \quad = \quad -\frac{1}{2\pi}\ln|x-y| + \frac{\mathrm{i}}{4} - \frac{1}{2\pi}\left(\ln k_- + C - \ln 2\right) + P_0(x,y),$$

where $C = 0.57721...$ *is the Euler constant and*

$$P_0(x,y) := G_0(x,y) + O(|x-y|\ln|x-y|)$$

approaches zero as $|x-y| \to 0$.

2.1 Computing B_{ij}

When computing the integrals B_{ij} which do not involve a logarithmic singularity then standard quadrature rules can be applied. Let us denote by Δ_j the area of the triangle K_j, and by $v_\ell^j, \ell = 1,2,3$ the vertices of K_j. Hence, when integrating $G(x_i,\cdot)$ over a triangle K_j for $j \neq i$, where x_i denotes the center of K_i, we may approximate B_{ij} using a simple rule as for example

$$B_{ij} \quad = \quad \frac{\Delta_j}{3}\sum_{\ell=1}^{3} G(x_i, v_\ell^j), \quad i \neq j.$$

On the other hand, a key observation is that the singularity of the integrand in B_{ii} at $y = x_i$ can be integrated exactly over K_i. Indeed, we split the integrand into a regular part

$$M_0(x_i,y) := G(x_i,y) + \frac{1}{2\pi}\ln|x_i - y|, \quad y \in K_i,$$

which can be integrated via quadrature and an integrable singular part such that

$$\begin{aligned} B_{ii} \quad &:= \quad -\frac{1}{2\pi}\int_{K_i}\ln|x_i-y|dy + \int_{K_i} M_0(x_i,y)dy \\ &\approx \quad -\frac{1}{2\pi}\int_{K_i}\ln|x_i-y|dy + \frac{\Delta_i}{3}\sum_{\ell=1}^{3} M_0(x_i, v_\ell^i) \end{aligned}$$

Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 5, N. 1, 2017.

2.2 Exact integration of $\ln|x|$ over a triangle centered at the origin O

By a shift of K_i to the origin we see that

$$I_i := \int_{K_i} \ln|x_i - y|dy \;=\; \int_{\hat{K}_i} \ln|y|dy,$$

where $\hat{K}_i := \{x - x_i : x \in K_i\}$ is now centered at the origin.

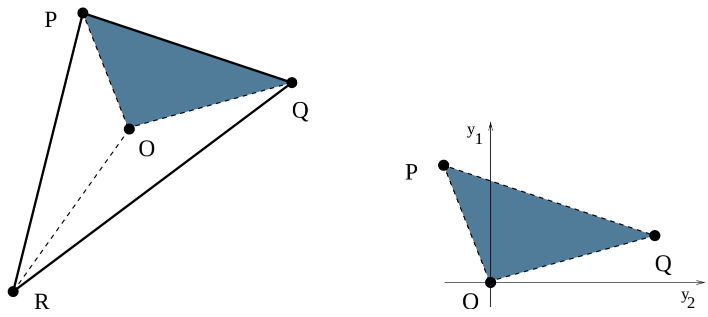

Figure 1: Split of $\hat{K}$ into three triangles.

Therefore, without loss of generality, we consider a non-degenerate generic triangle $\hat{K}$ with vertices P, Q and R, centered at the origin. If we also denote $\hat{K}$ by $\overline{PQR}$, we promptly notice that $\hat{K}$ is composed of three sub-triangles, denoted by $\overline{OPQ}$, $\overline{OPR}$ and $\overline{OQR}$ as shown in Fig. 1. The integral to be considered is

$$I := \int_{\hat{K}} \ln|y|dy,$$

which can be split into 3 integrals, each of them over a triangle consisting of the origin and 2 other vertices, that is,

$$I = \left(\int_{\overline{OPR}} + \int_{\overline{OPQ}} + \int_{\overline{OQR}}\right) \ln|y|dy.$$

Each integral can then be computed exactly by first re-mapping the triangle (with the origin as one vertex) of cartesian coordinates (y_1, y_2) into a canonical right triangle with local variables (s, t) and then using Duffy's transformation (u, ξ)

$$s = \xi, \quad t = \xi u,$$

which, in turn, allows exact integration. Fig. 2 shows the mapping of the unit triangle onto Duffy´s unit square.

By considering the triangle $\overline{OPQ}$ of area $\triangle_{OPQ} > 0$ where the cartesian coordinates of P, Q are identified with the vectors $\vec{p}$ and $\vec{q}$ [2] respectively. Then, one can show that

$$\int_{\overline{OPQ}} \ln|y|dy = \frac{\triangle_{OPQ}}{2}\left(-3 + \ln A + 2\,\Re e\left\{\ln(1-u_0)(1-u_0) + \ln(-u_0)u_0\right\}\right)$$

where $A := |\vec{q} - \vec{p}|^2$, $B := \vec{p} \cdot (\vec{q} - \vec{p})$, $C := |\vec{p}|^2$ and $u_0 := (-B + 2\mathrm{i}\triangle_{OPQ})/A$

[2] Here we locally change the notation $\vec{v}_1 \leftarrow \vec{p}$ and $\vec{v}_2 \leftarrow \vec{q}$

DOI: 10.5540/03.2017.005.01.0333

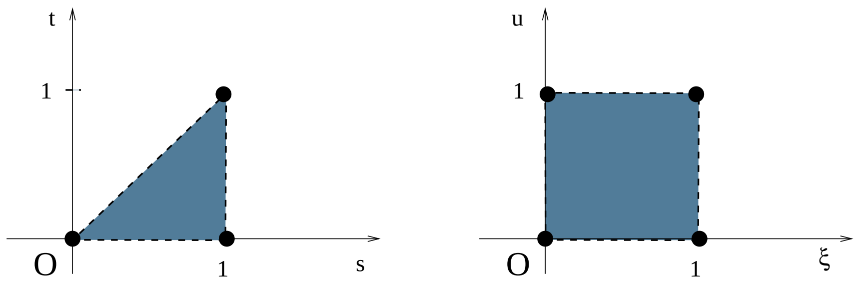

Figure 2: Canonical triangle σ mapped into Duffy's square

2.3 Scattered field

Let $x \notin D$. With (2) in mind the scattered field is such that

$$u^s(x) \;=\; -k_-^2 \int_D m(y)u(y)G(x,y)dy = -k_-^2 \int_D v(y)G(x,y)dy$$

which can be approximated by

$$u^s(x) \;=\; -k_-^2 \sum_j a_j \int_{K_j} G(x,y)dy \approx -k_-^2 \sum_j a_j \left(\frac{\Delta_j}{3} \sum_{\ell=1}^{3} G(x, v_\ell^j) \right)$$

or some other suitable quadrature.

3 Examples

We now qualitatively compare the internal total fields to the analytical solution when the background medium is homogeneous with $k_\pm = 6$ and the penetrable scatterer is a unitary disk centered at the origin with refractive index $n = 0.25$. The incident point source is located at $(0.5, 2)$ and the mesh we used is indicated in Fig. 3. The fields are indeed in agreement as we can see when comparing the collocation approximation in Fig. 4, with the analytical fields (see e.g. [3]) in Fig. 5.

The validation of the collocation method was also confirmed when we compared it to the BEM solution of a perfect conductor (PEC) cylinder with arbitrary cross section (developed independently) buried in a two-layered background. In the collocation solution we allowed a large conductivity, say $O(10^6)$, which *approximates* a perfect conductor. Within this framework and under frequencies in kHz of $O(1)$, the scattered fields were well approximated by the limiting PEC case.

The high-order formalism of this collocation scheme is as follows: if one uses piecewise continuous basis functions in such a way that in each element of the triangulation they consist of canonical polynomials of arbitrary order (as it is the case in traditional discontinuous Galerkin finite element schemes [4]), then we can show that the integrals involving the singularity can be evaluated *analytically* in a manner analogous to the lower order

DOI: 10.5540/03.2017.005.01.0333

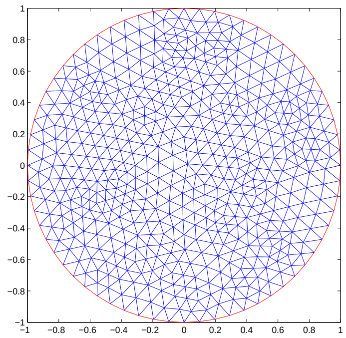

Figure 3: Mesh triangulation with 549 nodes and 1032 triangles

case, and we are left with a relatively straightforward high-order scheme which bypasses the singularity.

As usual in the devising and analysis of discontinuous elements schemes, one should investigate stability and how to eventually introduce the penalization of jumps across element interfaces, and from the viewpoint of wave propagation, one would be interested in a careful analysis of the behavior of the scheme in both the low (our test case) and high wavenumber regime [5, 6].

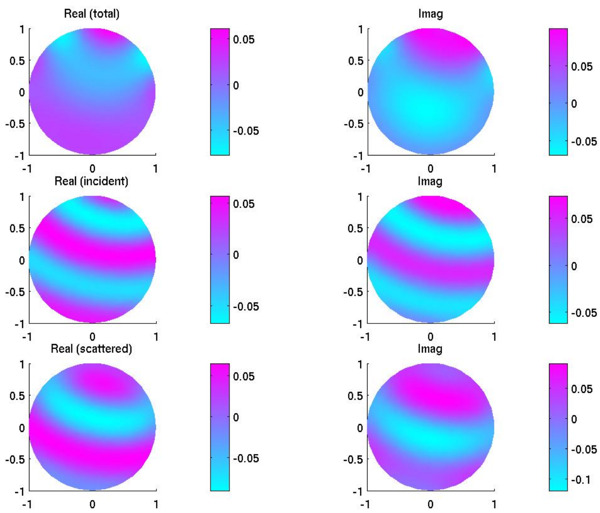

Figure 4: Internal total, incident and scattered (perturbation) field obtained by the collocation method

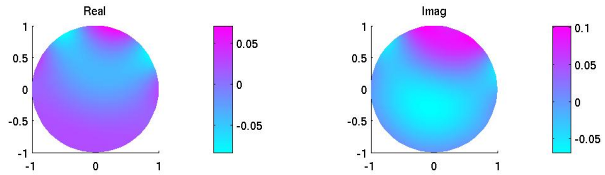

Figure 5: Analytical internal total field for the unit disk (to be compared to the top images in Fig. 4)

References

[1] G. Giorgi, M. Brignone, R. Aramini, and M. Piana. Application of the inhomogeneous Lippmann–Schwinger equation to inverse scattering problems. *SIAM J. on Appl. Math.*, 73(1):212–231, 2013.

[2] P. A. Martin. *Multiple scattering: interaction of time-harmonic waves with N obstacles.* Number 107. Cambridge University Press, 2006.

[3] J. J. Bowman, T. B. A. Senior, and P. L. E. Uslenghi, editors. *Electromagnetic and Acoustic Scattering by Simple Shapes.* Hemisphere Publishing Corp., New York, 1987. Revised reprint of the 1969 edition.

[4] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779 (electronic), 2001/02.

[5] M. Ainsworth, P. Monk, and W. Muniz. Dispersive and dissipative properties of discontinuous Galerkin finite element methods for the second order wave equation. *Journal of Scientific Computing*, 27(1-3):5–40, 2006.

[6] M. F. Wheeler. An elliptic collocation-finite element method with interior penalties. *SIAM J. Numer. Anal.*, 15(1):152–161, 1978.

