Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 5, N. 1, 2017.

Trabalho apresentado no CNMAC, Gramado - RS, 2016.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

On the (in)dependence of the Dedekind-Peano axioms for natural numbers

Márcia R. Cerioli[1]

Programa de Engenharia de Sistemas e Computação-COPPE e Instituto de Matemática, UFRJ, Rio de Janeiro, RJ

Hugo Nobrega[2]

Institute for Logic, Language, and Computation, University of Amsterdam, The Netherlands

Guilherme Silveira[3]

Instituto de Matemática, UFRJ, Rio de Janeiro, RJ

Petrucio Viana[4]

Instituto de Matemática e Estatística, UFF, Niterói, RJ

Abstract. We present a direct proof that the Dedekind-Peano axioms for the sequence of natural numbers are not completely independent, as well as a new completely independent set of axioms based on the same set of primitives as the one originally proposed by R. Dedekind.

Key-words. Peano Axioms, Weak independence, Strong independence, Syntactical proofs

1 Introduction

Natural numbers are among the most basic objects which play some role in the foundations of both Computer Science and Mathematics. Without going any deep on this matter, we just emphasize the role natural numbers play in the arithmetization of analysis and in the development of techniques to proofs of program termination.

As is very well known, a proper axiomatic foundation for the theory of natural numbers emerged from the works by Peirce [8], Dedekind [2], and Peano [7], which presented slightly different sets of axioms characterizing the sequence of natural numbers up to isomorphism.

Independence is one of the classical properties a set of axioms may satisfy, the others being consistency, satisfiability, completeness, and categoricity [15]. A lot of attention has been devoted to devising independent sets of axioms for various basic mathematical

[1]cerioli@cos.ufrj.br
[2]hugonobrega@globo.com
[3]guilhermelas@yahoo.com.br
[4]petrucio_viana@id.uff.br

theories, according to various notions of independence [1, 3, 9, 10, 13, 14]. In particular, since Peano, it is known that the axioms proposed by Dedekind are *independent* in the usual sense that none of them is a consequence of the others (cf. [5]). On the other hand, by means of algebraic tools, Henkin [4] proved that the Dedekind-Peano axioms are not *completely independent*, in the sense proposed by Moore [6] (cf. Section 3). Henkin left as an open problem that of providing a direct, purely logical proof of his result. In fact, before Henkin, Wang [14] had already proved that the Dedekind-Peano axioms are not completely independent, and proposed an alternative set of axioms for natural numbers which is completely independent, but the set of primitive concepts he used was different from that originally adopted by Dedekind and Peano.

In this note, we investigate the independence of the set of axioms for the sequence of natural numbers given by Dedekind [2] and Peano [7] a little bit further. After reviewing Peano's and Henki's results on independence, we contribute to this line of development by presenting a direct proof that the Dedekind-Peano axioms are not completely independent, as well as a new completely independent set of axioms based on the same set of primitives as the one originally used by Dedekind and Peano.

2 Weak (in)dependence of the Dedekind-Peano Axioms

In this section, we recall the Peano axioms and review their weak independence. All the results in this section are very easy and well known.

Definition 2.1. *Let* 0 *be a constant and* S *a unary function symbol. The* Dedekind-Peano axioms*, or simply DP axioms, are:*

Zer. $\forall x(Sx \neq 0)$.

Inj. $\forall x \forall y(Sx = Sy \to x = y)$.

Ind. $\forall X(0 \in X \wedge \forall x(x \in X \to Sx \in X) \to \forall x(x \in X))$.

A Dedekind-Peano structure*, or simply DP structure, is a structure* $\mathcal{N} = \langle N,\ 0^N,\ S^N \rangle$*, where* N *is a non-empty set,* $0^N \in N$*, and* $S^N : N \to N$.

A DP structure $\mathcal{N}$ *is a* Dedekind-Peano model*, or simply DP model, when the axioms* Zer*,* Inj*, and* Ind *are true in* $\mathcal{N}$*, when the symbols* 0 *and* S *are respectively interpreted as* 0^N *and* S^N.

The structure $\langle \mathbb{N},\ 0^{\mathbb{N}},\ S^{\mathbb{N}} \rangle$*, where* $\mathbb{N}$ *is the set of natural numbers,* $0^{\mathbb{N}}$ *is the number zero, and* $S^{\mathbb{N}}$ *is the usual successor function, is the* standard *DP model.*

Theorem 2.1. *The set* $\{\mathsf{Zer}, \mathsf{Inj}, \mathsf{Ind}\}$ *is satisfiable.*

Given a (satisfiable) set of axioms, there exist in the bibliography of foundations several forms of independence it may be asked to satisfy. Let Σ be a satisfiable set of axioms and φ be an axiom in Σ. The first notion we consider is the usual notion of independence, which is due to Peano (see, e.g., [5]).

Definition 2.2. *We say that* φ *is* weakly independent *from* Σ *when* $(\Sigma \setminus \{\varphi\}) \cup \{\neg\varphi\}$ *has a model.*

We say that Σ *is* weakly independent *when every* φ *in* Σ *is weakly independent from* Σ.

DOI: 10.5540/03.2017.005.01.0239

The following is also due to Peano [7].

Theorem 2.2. *The Dedekind-Peano axioms are weakly independent.*

3 Strong (in)dependence of the Dedekind-Peano Axioms

In this section, we investigate the strong independence of the DP-axioms. We recall Henkin's result showing throughout semantic means that the DP-axioms are not strongly independent. Besides, we present our main result: a completely syntactical proof that the DP-axioms are not strongly independent.

A direct generalization of weak independence was proposed by Moore [6].

Definition 3.1. *We say that* Σ *is* strongly independent *when* $(\Sigma \setminus \Gamma) \cup \{\neg\varphi : \varphi \in \Gamma\}$ *has a model, for every* $\Gamma \subseteq \Sigma$.

Observe that we obtain weak independence by restricting Definition 3.1 to the cases where Γ is an singleton subset of Σ.

Investigating the complete independence of the DP-axioms, we were unable to find a DP-structure in which $\neg\mathsf{Zer}$, $\neg\mathsf{Inj}$, and Ind were simultaneously true. The following result explain why this is impossible.

Theorem 3.1. *The Dedekind-Peano axioms are not completely independent.*

Theorem 3.1 is an immediate corollary of the following result.

Lemma 3.1. *Every model of* Ind *is a model of* $\mathsf{Inj} \vee \mathsf{Zer}$.

Proof. We present a sketch of the proof by Henkin [4] a more elaborated proof will be presented below.

In essence, the idea behind Henkin's proof of Lemma 3.1 [4] is the following. Let $\mathcal{N} = \langle N,\ 0,\ S\rangle$ be a DP structure such that Ind is true in $\mathcal{N}$. It then follows that $N = \{0, S0, SS0, \ldots\}$. Now, either $S^m0 \neq S^n0$ for all $m, n \in \mathbb{N}$, in which case we have that both Zer and Inj are true in $\mathcal{N}$, or there exists a least m such that $S^m0 = S^n0$ for some $n > m$. In this case, if $m = 0$ we have that $\neg\mathsf{Zer}$ and Inj are both true in $\mathcal{N}$, and if $m > 0$ we have that Zer and $\neg\mathsf{Inj}$ are both true in $\mathcal{N}$. □

Note that this proof makes fundamental use of the *numbers* $m, n, \ldots \in \mathbb{N}$ and of the well-ordering $\leq$ of $\mathbb{N}$ when talking about "the least m such that ...". So, in a certain sense, this is an *indirect* proof, by considering mathematical objects that are "out of the range" of its hypotheses. Henkin suggested the existence of a *direct* proof of the fact that every DP structure in which Ind is true is also a DP structure in which Zer and/or Inj is true, i.e., one "using only the laws of logic and the elements of set theory" [4, p. 324]. Considering ourselves to be what Henkin calls "enterprising readers", we have decided to take on his challenge and provide as simple a proof as we could of the following theorem.

Theorem 3.2. Inj *is a second-order syntactical consequence of* $\neg\mathsf{Zer}$ *and* Ind.

DOI: 10.5540/03.2017.005.01.0239

The proof proceeds in a series of results, based on the following abbreviations:

$$\begin{array}{rcl}\varphi_1(x,X) & \Leftrightarrow & 0 \in X \\ \varphi_2(x,X) & \Leftrightarrow & x \in X \\ \varphi_3(x,X) & \Leftrightarrow & \forall y \in N(y \neq x \wedge y \in X \rightarrow Sy \in X) \\ \varphi(x,X) & \Leftrightarrow & \varphi_1(x,X) \wedge \varphi_2(x,X) \wedge \varphi_3(x,X)\end{array}$$

In what follows, we work in a certain informal environment, but if the reader is concerned with the formalization of our results, everything we prove here can be developed from the usual (incomplete) set of formal axioms for second-order logic (cf. [12]). Due to lack of space, we left this important aspect of our work for a complete version of this paper. We also omit some of the proofs.

Lemma 3.2. *Let $\mathcal{N} = \langle N, 0, S\rangle$ be a DP-structure. Then, the following hold:*
(a) $\varphi(x, N)$.
(b) If $\varphi(x, X_i)$, for every $i \in I$, then $\varphi(x, \cap\mathcal{F})$.

Lemma 3.2 allows us to define

$$I_x \;=\; \bigcap\{X \subseteq N : \varphi(x,X)\},$$

for every $x \in N$. We immediately have $0 \in I_x$, $x \in I_x$, and $\varphi(x, I_x)$. In other words, I_x is the least subset of N that contains 0 and x as elements, and is closed under the operation S up to x.

Lemma 3.3. $I_0 = \{0\}$.

Lemma 3.4. $I_{Sx} \subseteq I_x \cup \{Sx\}$

Next, we define two binary relations on N. For every $x, y \in N$:

$$\begin{array}{rcl}xPy & \Leftrightarrow & I_x \subseteq I_y \\ xP^{\neq}y & \Leftrightarrow & xPy \wedge x \neq y\end{array}$$

The intended meaning of 'xPy ' is 'x is a predecessor of y'. It possesses, by definition, some properties an ordering relation on N must possesses. For example, we have the following immediate properties.

Lemma 3.5. *(a) xPx.*
(b) If xPy and yPz, then xPz.

But the reader must be cautioned that this relation may lacks some familiar properties that a proper ordering on N must possesses, for example, it may not agree with S, i.e. xPy may not imply $SxPSy$. Now, will take some effort to prove that P is a total ordering on N that possesses *some* of the familiar properties of $\leq$ on $\mathbb{N}$.

Lemma 3.6. $0Px$.

Lemma 3.7. $\neg(xP^{\neq}0)$.

DOI: 10.5540/03.2017.005.01.0239

Lemma 3.8. $I_x = \{y : yPx\}$.

Proof. To prove that $I_x \subseteq \{y : yPx\}$, it suffices to prove that $\varphi(x, \{y : yPx\})$. We have $\varphi_1(x, \{y : yPx\})$, because $0 \in \{y : yPx\}$, by lemma 3.6. We have $\varphi_2(x, \{y : yPx\})$, because $I_x \subseteq I_x$. Finally, we have $\varphi_3(x, \{y : yPx\})$, because taking $y \in N$ such that $y \neq x$ and $y \in \{y : yPx\}$, we have $I_y \subseteq I_x$. Since $y \neq x$, $\varphi_3(x, I_x)$ implies $Sy \in I_x$. Thus, $I_x \cup \{Sy\} \subseteq I_x$. Putting this together with Lemma 3.4, we have $I_{Sy} \subseteq I_y \cup \{Sy\} \subseteq I_x \cup \{Sy\} \subseteq I_x$. Whence, $Sy \in \{y : yPx\}$. To prove $\{y : yPx\} \subseteq I_x$, let $z \in \{y : yPx\}$. So, zPx, which, by definition, gives us $I_z \subseteq I_x$. Now, since zPz, we also have by definition, $z \in I_z$. So, $z \in I_x$. □

Lemma 3.9. $\varphi(x, \{y : \neg(xP^{\neq}y)\})$.

Proof. We consider two cases. If $x = 0$, take $y \in \{y : \neg(xP^{\neq}y)\}$. We have $\neg(0P^{\neq}y)$. Hence, $\neg(0Py)$ or $0 = y$. Hence, by lemma 3.6, $y = 0$. Thus $\{y : \neg(xP^{\neq}y)\} = \{0\}$. Since $\varphi(0, \{0\})$, the result follows. If $x \neq 0$, we proceed as follows. We have $\varphi_1(x, \{y : \neg(xP^{\neq}y)\})$, because $\neg(xP^{\neq}0)$, by lemma 3.7. We have $\varphi_2(x, \{y : \neg(xP^{\neq}y)\})$, because if we assume $xP^{\neq}x$ we would have $x \neq x$, a contradiction. We have $\varphi_3(x, \{y : \neg(xP^{\neq}y)\})$, because taking $y \in N$ such that $y \neq x$ and $y \in \{y : \neg(xP^{\neq}y)\}$, we have $\neg(xPy)$. Hence, by lemma 3.8, $x \notin I_y$. Now, we consider two cases. If $Sy = x$, then by definition of $P^{\neq}$, $\neg(xP^{\neq}Sy)$. If $Sy \neq x$, since $x \notin I_y$, we also have $x \notin I_y \cup \{Sy\}$. From this, by applying Lemma 3.4, we conclude $x \notin I_{Sy}$. Now, since $x \in I_x$, we have $I_x \not\subseteq I_{Sy}$ and, whence, $\neg(xP^{\neq}Sy)$. In both cases, we have $Sy \in \{y : \neg(xP^{\neq}y)\}$. □

From the previous lemmas we have the following corollary.

Corollary 3.1. *If xPy, then $\neg(yP^{\neq}x)$.*

Next lemma is the first time we make use of Ind.

Lemma 3.10. *If $\neg(xP^{\neq}Sx)$, then $\neg\exists y(xP^{\neq}y)$.*

Lemma 3.11. *If $y, z \in I_x$, then yPz or zPy.*

Proof. Define $\psi(x)$ to be the property $\forall\, y, z \in I_x(yPz$ or $zPy)$. We prove by induction on x that $\forall x\psi(x)$. We have $\psi(0)$, by lemmas 3.3 and 3.5(a). Suppose $\psi(x)$, that is, $\forall\, y, z \in I_x(yPz$ or $zPy)$. Let $y, z \in I_{Sx}$. We consider three cases. If $y = z = Sx$, by 3.5(a) $SxPSx$, so we have yPz. If $y \neq Sx$ and $z = Sx$, by the hypothesis and Lemma 3.8, $yPSx$ and $z = Sx$. So, we have yPz. If $y \neq Sx$ and $z \neq Sx$, by Lemma 3.4, $y, z \in Ix$. So, by the Induction Hypothesis, we have yPz or zPy. □

Lemma 3.12. *If $Sx = 0$, then $\forall y(y \in I_x)$.*

Lemma 3.13. *If $Sy = Sz$ and $\exists x(yP^{\neq}x \wedge zP^{\neq}x)$, then $y = z$.*

Theorem 3.3. *If $\exists x(Sx = 0)$, then $\forall y \forall z(Sy = Sz \rightarrow y = z)$.*

Proof. Let x be such that $Sx = 0$. So, by lemma 3.12, we have $\forall y(y \in I_x)$. We consider two cases. If $Sy = Sz = 0$, then by lemma 3.12 again, $I_x = I_y = I_z$. But, by lemma 3.11, P is a total order on I_x, whence we conclude $x = y = z$. If $Sy = Sz \neq 0$, then $x \neq y$ and $x \neq z$. But, by lemma 3.8, $I_x = \{u : uPx\}$, so we have $yP^{\neq}x$ and $zP^{\neq}x$. Now, from Lemma 3.13 we conclude $y = z$. □

DOI: 10.5540/03.2017.005.01.0239

4 A Completely Independent Set of Axioms

In this section, we show that a completely independent set of axioms written in the same language as, and very similar to, the Dedekind-Peano axioms can be obtained by adapting the induction axiom in a simple way.

Definition 4.1. *The* bi-induction axiom *is:*

$$\mathsf{Bind.}\ \forall X[0 \in X \wedge \forall x(x \in X \leftrightarrow Sx \in X) \rightarrow \forall y(y \in X)].$$

Theorem 4.1. *Axioms* Zer*,* Inj*, and* Bind *are satisfiable.*

Proof. Take the standard model. □

Theorem 4.2. *Axioms* Zer*,* Inj*, and* Bind *are completely independent.*

Proof. For each subset $\Gamma \subseteq \{\mathsf{Zer}, \mathsf{Inj}, \mathsf{Bind}\}$, we provide the smaller DP-structure in which $\{\neg\varphi : \varphi \in \Gamma\}$ together with the other remaining axioms are true. Of course, the biggest part of this table is just a repetition of what we already have in Section 2.

1. $\langle N,\, a,\, S^N\rangle$, where $N = \{a\}$ and $S^N a = a$, is a model of {¬Zer, Inj, Bind}.
2. $\langle N,\, a,\, S^N\rangle$, where $N = \{a, b\}$ and $S^N a = S^N b = b$, is a model of {Zer, ¬Inj, Bind}.
3. $\langle N,\, 0^{\mathbb{N}},\, S^N\rangle$, where $N = \mathbb{N} \cup \{\sqrt{2}\}$, $S^N(x) = S^{\mathbb{N}}x$ for all $x \in \mathbb{N}$, and $S^N(\sqrt{2}) = \sqrt{2}$, is a model of {Zer, Inj, ¬Bind}.
4. $\langle N,\, a,\, S^N\rangle$, where $N = \{a, b\}$ and $S^N(a) = S^N(b) = a$, is a model of {¬Zer, ¬Inj, Bind}.
5. $\langle N,\, a,\, S^N\rangle$, where $N = \{a, b\}$, $S^N(a) = a$, and $S^N(b) = b$, is a model of {¬Zer, Inj, ¬Bind}.
6. $\langle N,\, a,\, S^N\rangle$, where $N = \{a, b, c\}$, $S^N(a) = S^N(b) = b$, and $S^N(c) = c$, is a model of {Zer, ¬Inj, ¬Bind}.
7. $\langle N,\, a,\, S^N\rangle$, where $N = \{a, b, c\}$, $S^N(a) = S^N(b) = a$, and $S^N(c) = c$, is a model of {¬Zer, ¬Inj, ¬Bind}.

This concludes the proof. □

A slightly weaker version of Dedekind's Homomorphism Theorem [2] follows from axioms Zer, Inj, and Bind.

Theorem 4.3 (Weak Homomorphism theorem)**.** *If* $\mathcal{N} = \langle N,\, 0^N,\, S^N\rangle$ *is a model of* {Zer, Inj, Bind}*, then for any structure* $\mathcal{M} = \langle M,\, 0^M,\, S^M\rangle$ *that is a model of* Ind*, there exists a unique homomorphism from* $\mathcal{N}$ *into* $\mathcal{M}$*, i.e., a unique function* $\phi : N \rightarrow M$ *such that* $\phi 0^N = 0^M$ *and* $\phi S^N x = S^M \phi x$*.*

Moreover, as in Dedekind [2], we have the following corollaries.

DOI: 10.5540/03.2017.005.01.0239

Corollary 4.1 (Categoricity theorem)**.** *Any two models of* {Zer, Inj, Bind} *are isomorphic.*

Corollary 4.2 (Completeness theorem)**.** *If a sentence is true in* $\mathbb{N}$*, then it is a consequence of* {Zer, Inj, Bind}*.*

From the Categoricity Theorem, we have that the our axioms are indeed equivalent to DP axioms, and we are done.

Acknowledgements

This research is partially supported by CNPq. The first author's research was supported in part by CNPq and FAPERJ (Bolsa Nota 10), and the second author's research was supported in part by CNPq and FAPERJ.

References

[1] D. F. Dawson, On absolutely independent group axioms, *Monatsh. Math.*, 71:393–401, 1967.

[2] R. Dedekind, *Was sind und was sollen die Zahlen?*, Vieweg: Braunschweig, 1888.

[3] F. Harary, A very independent axiom system, *Am. Math. Mon.*, 68:159–162, 1961.

[4] L. Henkin, On mathematical induction, *Am. Math. Mon.*, 67:323–338, 1960.

[5] H. C. Kennedy, The origins of modern axiomatics: Pasch to Peano, *Am. Math. Mon.*, 79:133–136, 1972.

[6] E. H. Moore, *Introduction to a Form of General Analysis*, Yale University Press, 1910.

[7] G. Peano, *Arithmetices Principia, Novo Methodo Exposita*, Bocca, 1889.

[8] C. S. Peirce, On the Logic of Number, *Am. J. Math.*, 4:85–95, 1881.

[9] K. R. Popper, A set of independent axioms for probability, *Mind*, 47:275–277, 1938.

[10] J. F. Rigby and J. Wiegold, Independent axioms for vector spaces, *Math. Gaz.*, 57:56–62, 1973.

[11] P. Sally, The complete independence of some common axiom systems, *Am. Math. Mon.*, 69:794–797, 1962.

[12] D. Siekfes, Decidable Theories I, *Lecture Notes in Math.*, 120, Springer, 1970.

[13] S. Ucsnay, A note on Tarski's note, *Am. Math. Mon.*, 115:66–68, 2008.

[14] S. Wang, A system of completely independent axioms for the sequence of natural numbers, *J. Symbolic Logic*, 8:41–44, 1943.

[15] R. L. Wilder, *Introduction to the Foundations of Mathematics*, 2nd ed., John Wiley & Sons, 1965.

DOI: 10.5540/03.2017.005.01.0239