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q-Analogues of Jacobsthal Identities Via Weighted Tilings

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Abstract. Our primary goal in this work is to state and prove the $q-$analogues for Jacobsthal identities, using the combinatorial techniques involving weighted tilings.

Keywords: Weighted tilings, $q-$Analogues Identities, Jacobsthal Numbers.

1 Introduction

A combinatorial interpretation of the Pell numbers was introduced by Benjamim, Plott and Sellers in [1] and Brigss, Little and Sellers in [3], stated and proved $q-$analogues of several Pell identities via weighted tilings. Suppose that there are $a \geq 1$ different colors of squares, $s_1, s_2, \cdots s_a$, and $b \geq 1$ different colors of dominoes, $d_1, d_2, \cdots, d_b$. Let $w_q(t)$ be the $q-$weight of these colored tiles defined as:

$$w_q(t) = \begin{cases} q^{ij}, & \text{if } t \text{ is a } d_j \text{ colored domino at position } (i, i+1); \\ q^{i(j-1)}, & \text{if } t \text{ is an } s_j, \text{ colored square at position } i; \end{cases}$$

and a corresponding generating function for Pell tilings of a $n-$board as follows:

$$P_{n+1}(a, b; q) = \frac{1 - q^{a(n+1)}}{1 - q^{n+1}} P_n(a, b; q) + q^n \frac{1 - q^{bn}}{1 - q^n} P_{n-1}(a, b; q),$$

with inicial conditions $P_0(a, b; q) = 1, P_1(a, b; q) = \frac{1-q^a}{1-q}$.

The nth Jacobsthal number, denoted by a_n is defined recursively by $a_0 = 0, a_1 = 1$, and $a_n = a_{n-1} + 2a_{n-2}$, for all $n \geq 2$. Following the ideas of [1], a_n can be interpreted as the number of tilings of a $1 \times n$ board using white squares, black dominoes, and gray dominoes, called number of Jacobsthal tilings of lenght n. The nth Jacobsthal-Lucas number, denoted by j_n is defined recursively by $j_0 = 2, j_1 = 1$, and $j_n = a_{n+1} + 2a_{n-1}$, for all $n \geq 2$. The

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Jabosthal-Lucas number can be intepreted as the number of bracelets of a $1 \times n$ board using white squares, black dominoes, and gray dominoes.

For the purposes of this paper, we will focus on the following Jacobsthal identities presented in [4].

Theorem 1.1. *For all* $n \geq 0$,

$$a_n = \sum_{r \geq 0} \binom{n-r}{r} 2^r. \tag{1}$$

Theorem 1.2. *For all* $n \geq 0$,

$$2 \sum_{i=0}^{n} a_i = a_{n+2} - 1. \tag{2}$$

Theorem 1.3. *For all* $n \geq 0$,

$$a_{2n+1} = \sum_{i=0}^{n} 2^{n-i} a_{2i}. \tag{3}$$

Theorem 1.4. *For all* $n \geq 0$,

$$a_m a_{n+1} + 2 a_n a_{m-1} = a_{m+n+1}. \tag{4}$$

Theorem 1.5. *For all* $n \geq 1$,

$$a_n^2 = a_{n+1} a_{n-1} + (-1)^n 2^n. \tag{5}$$

Theorem 1.6. *For all* $n \geq 0$,

$$j_n = \sum_{r \geq 0} \frac{n}{n-r} \binom{n-r}{r} 2^r. \tag{6}$$

Theorem 1.7. *For all* $n \geq 0$,

$$2 \sum_{i=0}^{n} j_i = j_{n+2} - 1. \tag{7}$$

Theorem 1.8. *For all* $n \geq 0$,

$$j_{2n+1} = \sum_{i \geq 0} 2^{n-i} j_{2i}. \tag{8}$$

The main goal of this work is to state and prove the $q-$analogues for identities above, using the combinatorial techniques presented in [1], [2] and [3].

2 The q-Jacobsthal Numbers

The $q-$Jacobsthal numbers $J_n(q)$ are defined by

$$J_{n+1}(q) = J_n + (q + q^{2n})J_{n-1}; n \geq 1,$$

with inicial conditions $J_0(q) = J_1(q) = 1$. Clearly, the q-Jacobsthal number $J_n(q)$ coincides with the values a_n when $q = 1$. We define the weight of the tile t as follows:

$$w(t) = \begin{cases} i, & \text{if } t \text{ is a gray domino at position } (i, i+1), \\ 2i, & \text{if } t \text{ is a black domino at position } (i, i+1), \\ 0, & \text{if } t \text{ is a white square at position } i. \end{cases}$$

Let $\mathcal{T}_n$ be the set of all tilings of an n-board with white squares, black dominoes and gray dominoes. Then, for any tilings $T \in \mathcal{T}_n$ define the q-weight of T by

$$w_q(T) = \prod_{t \in T} q^{w(t)},$$

and define

$$\widetilde{J}_n(q) = \sum_{T \in \daleth_n} w_q(T).$$

Is easy to see that $J_n(q) = \widetilde{J}_n(q)$. For example with n=3, $\widetilde{J}_3(q) = 1 + q + 2q^2 + q^4 = J_2(q) + (q^2 + q^4)J_1(q) = J_3(q)$.

3 Some Analogues of q-Jacobsthal Identities

Given the definition of the $q-$Jacobsthal numbers $J_n(q)$ above, we now state the $q-$analogue of theorems 1.1 to 1.8 and prove some identities via these weighted tilings. We start defining the polynomial $p_{j,k,l}$ in q generated as coefficient of $x^j y^k z^l$ in the expansion of $(x + y + z)^{j+k+l}$ with inversions $yx = q^2xy, zx = q^2xz, zy = qyz$. For n=4, the contributions for the coefficient of xy^2z comes from the factors $(xy^2z + xyzy + xzy^2) + (yxyz + yxzy + y^2xz + y^2zx + yxyz + yxzy) + (zy^2x + zyxy + zxy^2)$ in the expansion of $(x + y + z)^4$. So using the inversions above we obtain

$$\begin{aligned} &\left((1 + q + q^2) + (q^2 + q^3 + q^4 + q^6 + q^2 + q^3) + (q^8 + q^6 + q^4)\right) xy^2z \\ &= \left(q^6 \frac{1 - q^3}{(1 - q)} + q\frac{1 - q^6}{(1 - q)} + \frac{1 - q^6}{(1 - q^2)}\right) xy^2z \\ &= p_{1,2,1}xy^2z. \end{aligned}$$

Let $\mathbb{T}_{j,k,l}$ be the set of tilings of n-board using exactly j black dominoes, k gray dominoes and l white squares, where $n = 2j + 2k + l$. For each $T \in \mathbb{T}_{j,k,l}$ we will associated a sequence, δ_T, replacing each black domino with an x, each gray domino with a y, and each white square with a z. This sequence is in the set $\mathbb{S}_{x^j,y^k,z^l}$ of all sequences with j characters equal to x, k characters equal to y and l characters equal to z. Thus, for each sequence in $\delta \in \mathbb{S}_{x^j,y^k,z^l}$ there is an associated tiling $T_\delta \in \mathbb{T}_{j,k,l}$.

Now, we start the process of computing the weight of generic tiling $T \in \mathbb{T}_{j,k,l}$. Firstly note that the tiling minimum weight, $T_{min} \in \mathbb{T}_{j,k,l}$, corresponds a sequence δ_{min} given by

$$\delta_{min} = \underbrace{xxx\cdots xx}_{j}\underbrace{yyy\cdots yy}_{k}\underbrace{zzz\cdots zz}_{l}.$$

This assertion is a consequence of following statements:

1. the weight of a black dominoes followed by a gray dominoes is less than that of a gray dominoes followed by a black dominoes.

2. the weight of a black dominoes followed by a white square is less than that of a white square followed by a black dominoes.

3. the weight of a gray dominoes followed by a white square is less than that of a white square followed by a gray dominoes.

Furthermore, the q-weight of T_{min} is given by

$$q^{2.\sum_{i=1}^{j} 2j-1+\sum_{m=1}^{k} 2j+2m-1} = q^{2j^2+2kj+k^2}.$$

We want to study the difference between the q-weight of generic tiling $T \in \mathbb{T}_{j,k,l}$ and the q-weight of minimum tiling T_{min}.

Given a sequence $\delta \in \mathbb{S}_{x^j,y^k,z^l}$ we consider, as before, the inversions $yx = q^2xy$, $zx = q^2xz$ and $zy = qyz$.

For example, the sequence y^2zxyx, is associated to the tiling $T \in \mathbb{T}_{2,3,1}$. In this case, we have T_{min} with minimum weight q^{29} associated to the sequence x^2y^3z.. Since

$$y^2zxyx = yyzxyx = q^2yyxzyx = q^3yyxyzx = q^5yyxyxz = q^7yyxxyz$$

$$= q^9yxyxyz = q^{11}yxxyyz = q^{13}xyxyyz = q^{15}x^2y^3z,$$

it follows that

$$w(T) = q^{15}w(T_{min}) = q^{15}q^{29} = q^{44}.$$

Then the weight $w(T)$ is given in terms of $w(T_{min})$ by a multiplication of a power of q.

Consider the sequence associate to the tiling of minimum weight q^{13}, x^2yz, which corresponds to the tiling with two black dominoes, one gray domino and one white square. Through the inversion of characters x, y and z, we obtain:

$xxyz = x^2yz; xxzy = qx^2yz; xyxz = q^2x^2yz; xzxy = q^3x^2yz; xyzx = q^4x^2yz; yxxz = q^4x^2yz; xzyx = q^5x^2yz; zxxy = q^5x^2yz; yxzx = q^6x^2yz; zxyx = q^7x^2yz; yzxx = q^8x^2yz; zyxx = q^9x^2yz$.

Thus, the polynomial $1 + q + q^2 + q^3 + 2q^4 + 2q^5 + q^6 + q^7 + q^8 + q^9$ corresponds to the sequence x^2yz. This polynomial tell us the power of q to be multiplied by the minimum weight. So,

$$(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + q^6 + q^7 + q^8 + q^9)q^{13}$$

is the weight generating function of tilings with two black dominoes, one gray domino and one white square.

Note that, in the expansion of the polynomial $(x + y + z)^4$, the coefficient of x^2yz is exactly the polynomial corresponding to the sequence x^2yz.

Furthermore, the polynomial $p_{j,k,l}$ generated as coefficient of $x^jy^kz^l$ in the expansion of $(x + y + z)^{k+j+l}$ determines the weight generating function of tilings with exactly j black dominoes, k gray dominoes and l white squares. Thus

$$\begin{aligned}\sum_{T\in\mathbb{T}_{k,j,l}} w_q(T) &= \sum_{\delta\in\mathbb{S}_{x^k,y^j,z^l}} w_q(T_\delta) \\ &= w(T_{min})p_{j,k,l} \\ &= q^{2j^2+2kj+k^2} p_{j,k,l}\end{aligned}$$

and we provide an important lemma.

Lemma 3.1. *The generating function for tilings with exactly j black dominoes, k gray dominoes and l white squares is given by*

$$q^{2j^2+2kj+k^2} p_{j,k,l}.$$

Now, we can prove the following q-analogue of Theorem (1.1).

Theorem 3.1. *(q-analogue of Theorem* (1.1) *) For all $n \geq 0$,*

$$J_n(q) = \sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} q^{r^2} \sum_{j=0}^{r} q^{j^2} p_{j,r-j,n-2r}.$$

Proof: The left-hand side q-counts the set of all Jacobsthal tilings of an n-board. Consider the tilings of n-board with exactly $r = j + k$ dominoes, j black dominoes and k gray dominoes. These tilings must have $n - 2r$ white squares. Applying Lemma (3.1) we obtain

$$q^{2j^2+2kj+k^2} p_{j,k,n-2r}.$$

By taking $k = r - j$, we obtain

$$q^{2j^2+2j(r-j)+(r-j)^2} p_{j,r-j,n-2r},$$

that is,

$$q^{r^2+j^2} p_{j,r-j,n-2r}.$$

Summing over all possible j and r

$$J_n(q) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=0}^{r} q^{r^2+j^2} p_{j,r-j,n-2r}.$$

as desired.

In the same way, is possible to determine an analogue of Theorem (1.6). To do this, we need define $J_n^m(q)$ as m-shifted tilings with exactly j black dominoes, k gray dominoes and l white squares, and then we obtain the following results.

Lemma 3.2. *The generating function for m-shifted tilings with exactly j black dominoes, k gray dominoes and l white squares is given by*

$$q^{m(2j+k)2j^2+2kj+k^2} p_{j,k,l}.$$

Theorem 3.2. *(q-analogue of Theorem* (1.6) *) For all* $n \geq 0$,

$$J_n(q) + (q^n + q^{2n})J_{n-2}^1(q) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} q^{r^2} \sum_{j=0}^{r} q^{j^2} \left[p_{j,r-j,n-2r} + q^{n-r+j+1} \times \right.$$

$$\left. \times \left\{ p_{j,r-j-1,n-2r} + q^{n+6j+1} p_{j-1,r-j,n-2r} \right\} \right].$$

Using the same techniques described above, we can state and prove $q-$analogues of theorems 1.2 through 1.8.

Theorem 3.3. *(q-analogue of Theorem* (1.2)*) For all* $n \geq 0$,

$$\sum_{i=0}^{n} (q^{i+1} + q^{2(i+1)}) J_i(q) = J_{n+2}(q) - 1. \tag{9}$$

Theorem 3.4. *(q-analogue of Theorem* (1.3)*) For all* $n \geq 0$,

$$J_{2n+1}(q) = \sum_{i=0}^{n} J_{2i}(q) \prod_{j=1}^{(n-i)} (q^{2(i+j)} + q^{4(i+j)}). \tag{10}$$

Theorem 3.5. *(q-analogue of Theorem* (1.4)*) For all* $n \geq 0$,

$$J_m(q) J_{n+1}^m(q) + (q^m + q^{2m}) J_n^{m+1}(q) J_{m-1}(q) = J_{m+n+1}(q). \tag{11}$$

Theorem 3.6. *(q-analogue of Theorem* (1.5)*) For all* $n \geq 1$,

$$(J_n(q))^2 = \begin{cases} J_{n+1}(q)J_{n-1}(q) - \left(q^{2p-1} + q^{2(2p-1)}\right) \prod_{j=1}^{p-1} \left(q^{2j-1} + q^{2(2j-1)}\right)^2, & \text{if n=2p-1} \\ J_{n+1}(q)J_{n-1}(q) + \prod_{j=1}^{p} \left(q^{2j-1} + q^{2(2j-1)}\right)^2, & \text{if n=2p.} \end{cases} \tag{12}$$

Theorem 3.7. *(q-analogue of Theorem (1.7)) For all $n \geq 1$,*

$$J_{n+2}(q) + (q^{n+2} + q^{2(n+2)})J_n^1(q) - 1 = (q^{n+2} + q^{2(n+2)}) \left\{ 1 + \sum_{k=2}^{n} (q^k + q^{2k}) J_{k-1}^1(q) \right\} + \sum_{k=0}^{n} (q^{k+1} + q^{2(k+1)}) J_k(q). \tag{13}$$

Theorem 3.8. *(q-analogue of Theorem (1.8)) For all $n \geq 1$,*

$$J_{2n+1}(q) + (q^{2n+1} + q^{2(2n+1)})J_{2n}^1(q) = J_0(q) \prod_{j=1}^{n} (q^{2j} + q^{4j}) + \sum_{i=1}^{n} \left\{ J_{2i}(q) \prod_{j=1}^{n-i} (q^{2i+2j} + q^{2(2i+2j)}) + (q^{2n+1} + q^{2(2n+1)}) J_{2i-1}^1(q) \prod_{j=1}^{n-i} (q^{2i+2j-1} + q^{2(2i+2j-1)}) \right\}.$$

Referências

[1] A. T. Benjamin, S. S. Plott and J.A. Sellers, Tiling Proofs of Recent Sum Identities Involving Pell Numbers *Annals of Combinatoric,* 12(3), 271-278 (2008)

[2] A. T. Benjamim and J. J. Quinn, Proofs that Really Count: The Art of Combinatorial Proof, *The Dolciani Mathematical Expositions, 27, Mathematical Association of America,* Washington, DC, 2003.

[3] K. S. Briggs, D. P. Little and J.A Sellers, Combinatorial Proofs of Various q-Pell Identities via Tilings *Annals of Combinatoric,* 14, 407-418 (2010)

[4] A. F. Horadam, Jacobsthal Representation Numbers, *The Fibonacci Quarterly* 34.1 (1996): 40-54.

[5] N. J. A. Sloane, The On-line Encyclopedia of Integer Sequences, *published eletronically at https://oeis.org/.*

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