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# A strange head-tail problem: a base-excited stick-slip oscillator

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**Abstract**. In this paper the dynamics of a dry-friction oscillator excited by a stochastic base motion is studied. The non-smooth behavior of the dry-frictional force associated with the non-smooth stochastic base motion induces in the system stochastic stick-slip oscillations. The focus of the paper is to analyze the stick-slip dynamics of the system from a probabilistic view point. Defined a time interval for analysis, the variables of interest are the number of time intervals in which stick or slip occur, the instants at which they begin, and their durations. These variables are modeled as stochastic objects and Monte Carlo simulations are employed to compute their statistics.

**Keywords**. Friction-induced vibration, stick-slip, non-smooth system, stick duration, stochastic quantification.

## 1 Introduction

Usually, stick-slip oscillations appear in mechanical systems in which uncertainties play an important role. For example in drilling. Some of sources of uncertainties are the bit-rock interaction, the presence of impacts, and fluid-structure interaction. In gears, randomness arises from manufacturing, assembly errors, and random load. Beyond these sources of uncertainties, the dry friction force itself presents an inherent random behavior [1, 2]. Because of this, a stochastic approach is the ideal way to address the problem of dry-friction [3–6]. The objective of the paper is to characterize, from a a probabilistic view point, the system response of a dry-friction oscillator excited by a stochastic base motion. The variables of interest are the number of time intervals in which stick occur, the instants at which they begin and their duration. These variables are modeled as stochastic objects. The focus of the paper is to compute statistics of them to give complete histograms, instead of the computation of some moments as is normally done.

## 2 Dynamics of the stick-slip oscillator

The system analyzed in this paper is composed by a simple oscillator (mass-spring) moving on a rough surface, as sketched in Fig. 1. The friction between the mass and the

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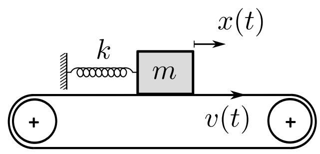


Figure 1: Stick-slip oscillator.

surface is modeled as Coulomb friction. Thus, the resulting motion of the mass can be characterized in two qualitatively different types, called modes: stick-mode (the mass and base have the same velocity during an open time interval) and the slip-mode (the mass and base have different velocities). The equation of motion of the system is [7]

$$m\,\ddot{x}(t) + k\,x(t) = f(t)\,, \tag{1}$$

where $x$ is the position of the mass over the base, $m$ is the mass, $k$ is the spring stiffness and $f$ is the frictional force between mass and base. During the slip-mode, $f(t) = n\mu\mathrm{sgn}(v(t) - \dot{x}(t))$, where $v$ is the base speed, $n$ is the normal force exercised by the base on the mass and $\mu$ is the constant friction coefficient. Besides this, the absolute value of the frictional force is equal to the maximum friction force, $f_{\max} = \mu n$. During the stick-mode, Eq. (1) can be rewritten as $m\dot{v} + k\,x(t) = f(t)$. The value of the frictional force during the stick-mode is confined to the interval $-f_{\max} \leq f \leq f_{\max}$. Then, once in a stick-mode, the mass stays moving with the base until $x(t) = \frac{n\mu}{k}$ in case of positive base velocity, or until $x(t) = -\frac{n\mu}{k}$ in case of negative base velocity. Observe that during the stick-mode, the modulus of the elastic force increases up to the limit value $|f_{\max}|$, i.e. the modulus of maximum friction force. When it exceeds this value, the stick-mode ends and the mass will start a slip-mode. Because of this, considering that base speed is constant in time, knowing the mass position when a stick starts, it is possible to predict its duration. Remark that the duration of the stick-mode is limited and its maximum value is $d_{\max} = 2\frac{n\mu}{k\,v}$.

# 3 Stochastic model of the base motion

The velocity of the base is modeled as a random process in time, constant by parts, $\mathcal{V}$. We consider that $\mathcal{V}$ assumes only the two values $1,0$ m/s and $-1,0$ m/s. Beside this, defined an interval $[0\,,t_a]$ for analysis, the number of changes of the velocity sign of $\mathcal{V}$ in this interval is given by a random variable $Q$ with Poisson distribution with parameter $\lambda t_a$. Thus, for $q = 0,\,1,\,2,\,\ldots$, the probability mass function of $Q$ is given by

$$\Pr(Q = q) = \frac{(\lambda t_a)^q\,e^{-\lambda t_a}}{q!}\,, \tag{2}$$

where $\lambda t_a$ is the mean and $\lambda$ is the expected value of number of changes per unit of time. Note that, so far, nothing was said about the instants in which the changes occur. To determine them, we use the fact that we want our random process $\mathcal{V}$ to be stationary and to be the most uncertain as possible. Therefore, it is reasonable to distribute all instants of changes over the interval $[0\,,t_a]$ in a completely arbitrary way. Then, these instants of changes are modeled as independent and identically distributed random variables, $P_1, P_2, \cdots, P_Q$, each of them uniformed distributed over $[0\,, t_a]$. Given a realization of $P_1, P_2, \cdots, P_Q$, i.e. given the $q$-uple $(p_1, p_2, \cdots, p_q)$, to generate a realization of $\mathcal{V}$ on $[0\,, t_a]$, we need to sort the samples. We transform the $q$-uple $(p_1, p_2, \cdots, p_q)$ in $(y_1, y_2, \cdots, y_q)$ in a way that $y_1 \leq y_2 \leq \cdots \leq y_n$. This operation generates new random variables, $Y_1, Y_2, \cdots, Y_Q$, wherein $Y1 = \min_{1 \leq i \leq n} \{P_1, \cdots, P_Q\}$. Due to the bang-bang base motion, if the mass is in the stick-mode in the instant just before the discontinuity on the base velocity, it must be in the slip-mode in the instant just after the discontinuity. Thus, the stick is interrupted by the discontinuities on the base velocity, as if the dynamics were reinitialized; all previous information is lost.

## 4 Statistical analysis fo the stick-slip process

As it was assumed that the base motion is uncertain, the response of the stochastic stick-slip oscillator is a random process. Defined a time interval for analysis, the variables of interest are the number of time intervals in which stick or slip occur, the instants at which they start, and their duration. They are modeled as stochastic objects, random variables or random processes.

- The number of time intervals in which stick occur is a random variable $S_T$.
- The number of time intervals in which slip occur is a random variable $S_L$.
- The instants at which the sticks begin are modeled by a discrete random process $\{T_1, \cdots, T_{S_T}\}$, where the subscripts $1, \cdots, S_T$ indicate the order that they occur, i.e., the instant in which starts the first stick, the second, and so on up to the $S_T$-th stick.
- The duration of the sticks are modeled by a discrete random process $\{D_1, \cdots, D_{S_T}\}$, where again the subscripts $1, \cdots, S_T$ indicate the order that they occur.
- The instants at which the slips begin are modeled by a discrete random process $\{L_1, \cdots, L_{S_L}\}$, where $1, \cdots, S_L$ indicate the order that they occur.
- The duration of the sticks are modeled by a discrete random process $\{H_1, \cdots, H_{S_L}\}$, where $1, \cdots, S_L$ indicate the order that they occur.

Figure 2 shows a sketch of the sequence of sticks and slips in the system response. Observe that we count the first slip just after the first stick, i.e., we have $L_1 > T_1$. Besides this, if the system response ends during a slip, the number of sticks is equal or the number of slips, i.e. $S_T = S_L$. If the system response ends during a stick, then $S_T = S_L + 1$ . Statistics of the stick-slip process were estimated by the Monte Carlo simulation method using $18,000$ independent realizations of random the system response [8, 9]. A previous convergence study was developed to determine the acceptable number of realizations. For computation, the duration $t_a$ was chosen as 10 seconds and $\lambda = 5.0$. For the integration, after some experiments with others methods, it was used the function *ode45* of the *Matlab*

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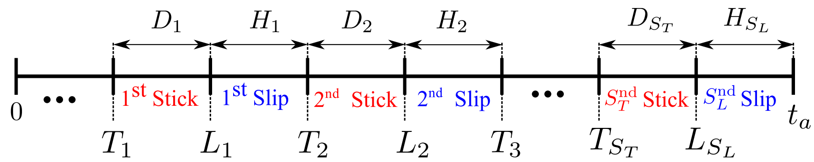
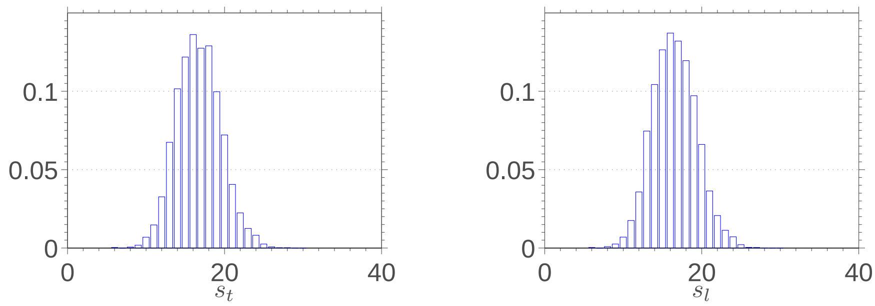



Figure 2: Sketch of the sequence of sticks and slips in the system response for the case in which $S_T = S_L$.

software, which applies the Runge-Kutta 4th/5th-order method as time-integration scheme with a varying time-step algorithm. The maximal step size is equal to $10^{-4}$ seconds, and the relative and absolute tolerance are equal to $10^{-9}$. The values of the parameters used in all simulations were 1.0 Kg for the mass, 4.0 N/m for the spring stiffness, 1.0 N for the normal force, 5.0 for the constant friction coefficient and $v_0 = 1.0$ m/s for the modulus of the base speed. The initial conditions of the system were modeled as two independent random variables, uniformed distributed over $[-1, 1]$. Figures 3(a) and 3(b) shows the normalized histograms of the number of time intervals in which stick and slip occur, i.e, the random variables $S_T$ and $S_L$. The estimated mean and variance to $S_T$ are respectively 16.71 and 7.94. To $S_L$ these values are 16.54 and 7.89.




(a) (b)

Figure 3: Normalized histograms constructed with 18,000 samples of (a) number of time-intervals in which stick occur and (b)number of time-intervals in which slip occur.

Figures 4 and 5 shows the normalized histograms of the first six instants at which sticks and slips begin, i.e., $T_1, \cdots, T_6$ and $L_1, \cdots, L_6$. Observing them we verify that $T_1, \cdots, T_6$ and $L_1, \cdots, L_6$ are not identically distributed random variables. As, $T_1 < \cdots < T_{S_T}$ and $L_1 < \cdots < L_{S_L}$, we have that $T_1, \cdots, T_{S_T}$ and $L_1, \cdots, L_{S_L}$ are not independent. Figure 6 shows the normalized histograms of the duration of the first six sticks, $D_1, \cdots, D_6$. The similarity between them suggests that $D_1, \cdots, D_{S_T}$ are identically distributed random variables. The estimated mean to these variables is 0.2 seconds. Figure 7 show the histograms of the duration of the first six slips, $H_1, \cdots, H_6$ and, again the similarity between them suggests that $H_1, \cdots, H_{S_L}$ are identically distributed random variables. The estimated mean to these variables is 0.4 seconds.

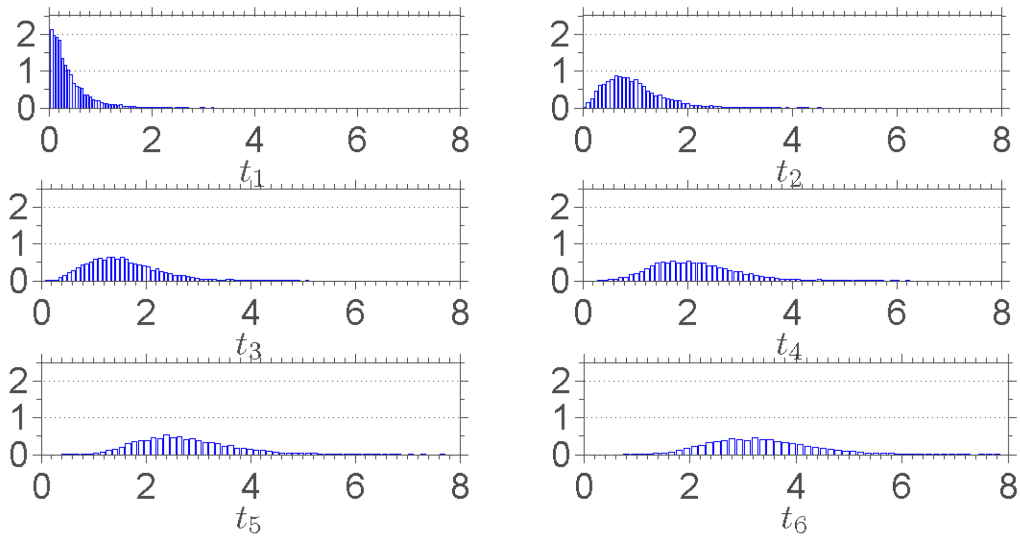



Figure 4: Normalized histograms constructed with 18,000 samples of the first six instants at which the sticks begin, $T_1, \cdots, T_6$

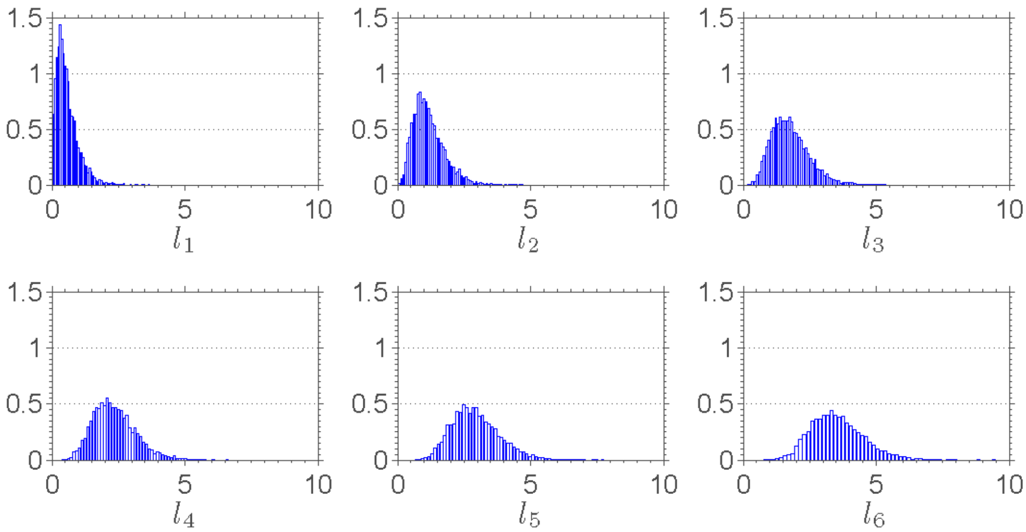



Figure 5: Normalized histograms constructed with 18,000 samples of the first six instants at which the slips begin, $L_1, \cdots, L_6$

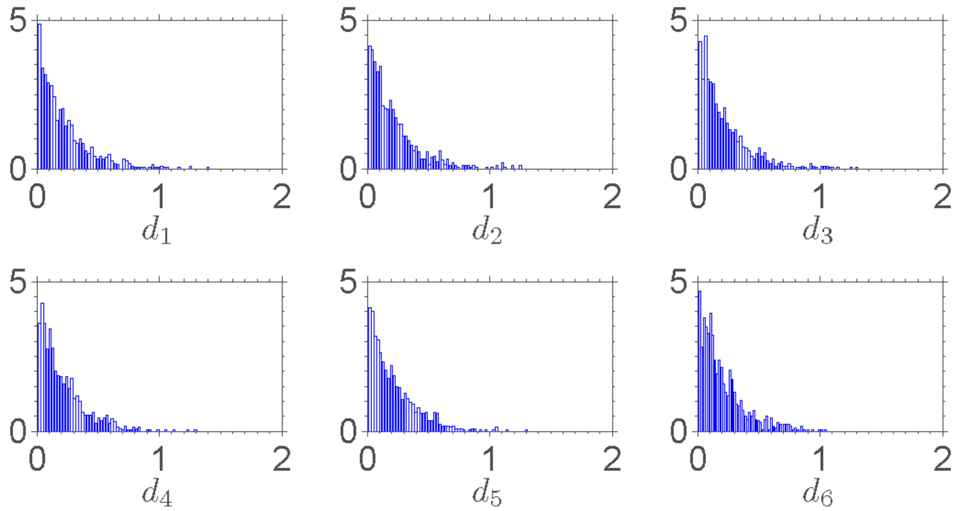



Figure 6: Normalized histograms constructed with 18,000 samples of the duration of the first six sticks, i.e., random variables $D_1, \cdots, D_6$.

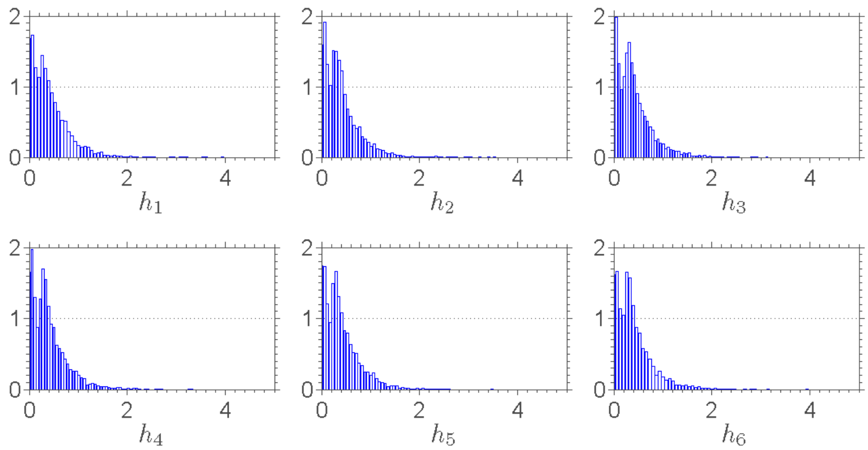



Figure 7: Normalized histograms constructed with 18,000 samples of the duration of the first six slips, i.e., random variables $H_1, \cdots, H_6$.

## 5 Conclusions

Considering that the average of the number of sticks is $\hat{\mu}_{S_T} = 16.71$ and the average of the stick duration is $\hat{\mu}_D = 0.20$ seconds, we compute the average of the total stick duration as $\hat{\mu}_{S_T} \times \hat{\mu}_D = 3.34$. This value represents one third of the the duration $t_a$. The the average of the total slip duration is two thirds of the the duration $t_a$. The simulations performed in the paper considered just one value to the friction coefficient, $\mu$, and to the parameter $\lambda$. To make a more complete analysis, we consider that it is important to verify the influence of $\lambda$ and $\mu$. This is an ongoing work, and there are still many investigation to perform. For instance, a robust optimization to maximize the performance of the drilling process considering the uncertainties in the friction model.

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## References

[1] Q. Feng. A discrete model of a stochastic friction system. *Computer Methods in Applied Mechanics and Engineering*, 192:2339 – 2354, 2003.

[2] M.T. Bengisu and A. Akay. Stick-slip oscillations: dynamics of friction and surface roughness. *Journal of the Acoustical Society of America*, 105(1):194–205, 1999.

[3] R. Lima and R. Sampaio. Stick-mode duration of random dry-friction oscillators. In *8th European Nonlinear Dynamics Conference (ENOC 2014)*, Vienna, Austria, 2014.

[4] R. Lima and R. Sampaio. A stochastic analysis of the trajectories of a dry-friction oscillator in a belt: how much sticked and slipping? In *Conferência Brasileira de Dinâmica, Controle e Aplicações (DINCON 2015)*, Natal, Brasil, 2015.

[5] R. Lima and R. Sampaio. Stick-mode duration of a dry-friction oscillator with an uncertain model. *Journal of Sound and Vibration*, 353:259–271, 2015.

[6] R. Lima and R. Sampaio. Analysis of a dry-friction oscillator driven by a stochastic base motion. In *Third International Symposium on Uncertainty Quantification and Stochastic Modeling (Uncertainties 2016)*, Maresias, Brazil, 2016.

[7] D. Jordan and P. Smith. *Nonlinear Ordinary Differential Equations*, volume 560. Oxford University Press, 2007.

[8] R. Sampaio and R. Lima. *Modelagem Estocástica e Geração de Amostras de Variáveis e Vetores Aleatórios*, volume 70 of *Notas de Matemática Aplicada*. SBMAC, http://www.sbmac.org.br/arquivos/notas/livro_70.pdf, 2012.

[9] E. Souza de Cursi and R. Sampaio. *Uncertainty Quantification and Stochastic Modeling with Matlab*. Elsevier, ISTE Press, 2015.

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