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On the properties of polynomials generated by a four-term recurrence relations

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In this work we consider the sequence of polynomials $\{P_n(z)\}_{n=0}^{\infty}$ generated by a four-term recurrence relation

$$P_n(z) + C(z)P_{n-1}(z) + B(z)P_{n-2}(z) + A(z)P_{n-3}(z) = 0, \tag{1}$$

where $A(z)$, $B(z)$ and $C(z)$ are linear polynomial in z, $P_0(z) = 1$ and $P_{-n}(z) = 0$, for all $n \in \mathbb{N}$.

Sequences of polynomials generated by recurrence relations have been many explored since the 18th century. A classical example is the three-term recurrence relation related to the real orthogonal polynomial sequence, which is a necessary condition for a sequence of polynomials to be orthogonal. More details, including the properties of the zeros (which are real), we may found in [2, 3]. However, in the case of the polynomials generated by a four-term recurrence relations of kind (1), not much is known. We can cite the references [1, 4, 5] as an example of recent studies involving these classes of polynomials. In common, these references present necessary and sufficient conditions on the coefficients of recurrence relation (1) for which all the zeros of $P_n(z)$ are real, i.e., the polynomial $P_n(z)$ is hyperbolic.

In particular, the authors in [4] presented a study of the behaviour of the zeros of polynomial $P_n(z)$ generated by relation (1) such that $A(z) = z$, $B(z) = b$ and $C(z) = c$ $(b, c \in \mathbb{R})$, with the initial conditions $P_0(z) = 1$, $P_1(z) = -c$ and $P_2(z) = -b + c^2$.

In the same way, in [5] the authors studies the zero location of the polynomials of the sequence $\{P_n(z)\}_{n=0}^{\infty}$ which satisfies (1) and $A(z) = a_0 + a_1 z$, $B(z) = b_0 + b_1 z$ and $C(z) = c$, where $a_0, a_1, b_0, b_1, c \in \mathbb{R}$.

In a recent contribution of [1], the author presented necessary and sufficient conditions so that $P_n(z)$ is hyperbolic, where $P_n(z)$ is a polynomial of sequence $\{P_n(z)\}_{n=0}^{\infty}$ generated by a four-term recurrence relation (1), where $A(z) = az$, $B(z) = b$ and $C(z) = cz$, with $a, b, c \in \mathbb{R}$.

Motivated by the results presented in [1, 4, 5], in this work we present some properties of the polynomials generated by a four-term recurrence relations

$$P_n(z) + (c_0 + c_1 z)P_{n-1}(z) + bzP_{n-2}(z) + azP_{n-3}(z) = 0, \tag{2}$$

with $a, b, c_0, c_1 \in \mathbb{R} - \{0\}$, $P_0(z) = 1$ and $P_{-n}(z) = 0$, for all $n \in \mathbb{N}$.

Based on the techniques presented in [4, 5], we will find necessary and sufficient conditions for $a, b, c_0, c_1 \in \mathbb{R} - \{0\}$ so that all the zeros of $P_n(z)$ are real. Moreover, we will present some properties of the polynomials $P_n(z)$ generated by (2). For example, some of them are easily shown: the degree of $P_n(z)$ is n and for any $n \geq 1$, the polynomials $P_n(z)$, $P_{n+1}(z)$ and $P_{n+2}(z)$ not have common zeros.

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