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Geodesics and Constant Angular Momentum in the de Sitter Manifold

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Abstract. Let $M = SO(1,4)/SO(1,3) \simeq S^3 \times \mathbb{R}$ (a parallelizable manifold) be a submanifold in the structure $(\mathring{M}, \mathring{\boldsymbol{g}})$ (hereafter called the bulk) where $\mathring{M} \simeq \mathbb{R}^5$ and $\mathring{\boldsymbol{g}}$ is a pseudo Euclidian metric of signature $(1,4)$. Let $\boldsymbol{i} : M \to \mathbb{R}^5$ be the inclusion map and let $\boldsymbol{g} = \boldsymbol{i}^* \mathring{\boldsymbol{g}}$ be the pullback metric on M. It has signature $(1,3)$. Let $\boldsymbol{D}$ be the Levi-Civita connection of $\boldsymbol{g}$. We call the structure $(M, \boldsymbol{g})$ a de Sitter manifold and $M^{dSL} = (M = \mathbb{R} \times S^3, \boldsymbol{g}, \boldsymbol{D}, \tau_{\boldsymbol{g}}, \uparrow)$ a de Sitter spacetime structure, which is of course orientable by $\tau_{\boldsymbol{g}} \in \sec \bigwedge^4 T^*M$ and time orientable (by $\uparrow$). Under these conditions, here we want to present the results that appears in [5–7] in particular that if the motion of a free particle moving on M happens with constant *bulk* angular momentum then its motion in the structure M^{dSL} is a timelike geodesic. Also any geodesic motion in the structure M^{dSL} implies that the particle has constant angular momentum in the bulk. So using the Clifford and spin-Clifford formalisms [3] and the natural hypothesis that a particle moving freely in $(M, \boldsymbol{g})$ has constant bulk angular momentum leads naturally to the Dirac equation as found in [1] in the de Sitter structure $(M, \boldsymbol{g})$.

Keywords. de Sitter Manifold, Geodesics, Angular Momentum, General Relativity

1 Introduction

In what follows $SO(1,4)$ and $SO(1,3)$ denote the special pseudo-orthogonal groups in $\mathbb{R}^{1,4} = (\mathring{M} = \mathbb{R}^5, \mathring{\boldsymbol{g}})$ where $\mathring{\boldsymbol{g}}$ is a metric of signature $(1,4)$. The *de Sitter manifold.*M can be viewed as a brane (a submanifold) in the structure $\mathbb{R}^{1,4}$. The structure $M^{dSL} = (M, \boldsymbol{g}, \boldsymbol{D}, \tau_{\boldsymbol{g}}, \uparrow)$ will be called *Lorentzian de Sitter spacetime structure* where, if $\boldsymbol{\iota} : \mathbb{R} \times S^3 \to \mathbb{R}^5$ is the inclusion mapping, $\boldsymbol{g} = \boldsymbol{\iota}^* \mathring{\boldsymbol{g}}$ and $\boldsymbol{D}$ is the parallel projection on M of the pseudo Euclidian metric compatible connection $\mathring{\boldsymbol{D}}$ in $\mathbb{R}^{1,4}$ (details in [4, 5]). As well known, $(M, \boldsymbol{g})$, a pseudo-sphere is a spacetime of constant Riemannian curvature. It has ten Killing vector fields. The Killing vector fields are the generators of infinitesimal actions of the group $SO(1,4)$ (called the de Sitter group) in M. The group $SO(1,4)$ acts transitively in $SO(1,4)/SO(1,3)$, which is thus a homogeneous space (for $SO(1,4)$).

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The structure M^{dSL} has been used by many physicists as an alternative arena for the motion of particles and fields in place of the Minkowski spacetime structure[3] $\mathfrak{M}$. One of the reasons is that the isometry group of the structure $(M, \boldsymbol{g})$ is the de Sitter group, which as well known reduces to the Poincaré group when he radius ℓ of $(M, \boldsymbol{g})$ goes to ∞. Now, as well known the natural motion of a free particle of mass m in $\mathfrak{M}$ occurs with constant momentum $\boldsymbol{p} = m\varkappa_*$ where $\varkappa : \mathbb{R} \to \mathcal{M}$ is a timelike curve pointing to the future. The question which naturally arises is the following:

Which is the natural motion of a free particle of mass m in the structure (M, g)?

One natural suggestion given the well known relation between the de Sitter and Poincaré groups [2] is that such a motion occurs with constant angular momentum $\boldsymbol{L}$ as determined by (hyper observers) living in the bulk. Given this hypothesis we proved in [6] the following proposition: (a)*:* If a particle travels with geodesic motion in the structure M^{dSL} then its bulk angular momentum $\boldsymbol{L}$ is constant. (b)*:* Also,if a particle of mass m constrained to move in M occurs with constant bulk angular $\boldsymbol{L}$ then its motion for an observer living in the brane M is described by a timelike geodesic in the structure M^{dSL}.

2 The Lorentzian de Sitter M^{dSL} Structure and its (Projective) Conformal Representation

Let $SO(1,4)$ and $SO(1,3)$ be respectively the special pseudo-orthogonal groups in the structures $\mathbb{R}^{1,4} = \{\mathring{M} = \mathbb{R}^5, \mathring{\boldsymbol{g}}\}$ and $\mathbb{R}^{1,3} = \{\mathbb{R}^4, \boldsymbol{\eta}\}$ where $\mathring{\boldsymbol{g}}$ is a metric of signature $(1,4)$ and $\boldsymbol{\eta}$ a metric of signature $(1,3)$. The manifold $M = SO(1,4)/SO(1,3)$ will be called the *de Sitter manifold.* Since

$$M = SO(1,4)/SO(1,3) \approx SO(1,4)/SO(1,3) \approx \mathbb{R} \times S^3 \tag{1}$$

this manifold can be viewed as a brane [4] (a submanifold) in the structure $\mathbb{R}^{1,4}$. In General Relativity studies it is introduced a Lorentzian spacetime, i.e., the structure $M^{dSL} = (M = \mathbb{R} \times S^3, \boldsymbol{g}, \boldsymbol{D}, \tau_{\boldsymbol{g}}, \uparrow)$ called *Lorentzian de Sitter spacetime structure*[4] where if $\boldsymbol{\iota} : \mathbb{R} \times S^3 \to \mathbb{R}^5$ is the inclusion mapping, $\boldsymbol{g} := \iota^* \mathring{\boldsymbol{g}}$ and $\boldsymbol{D}$ is the parallel projection on M of the pseudo Euclidian metric compatible connection in $\mathbb{R}^{1,4}$ (details in [5]). As well known, M^{dSL} is a spacetime of constant Riemannian curvature. It has ten Killing vector fields. The Killing vector fields are the generators of infinitesimal actions of the group $SO(1,4)$ (called the de Sitter group) in $M = \mathbb{R} \times S^3 \approx SO(1,4)/SO(1,3)$. The group $SO(1,4)$ acts transitively in $SO(1,4)/SO(1,3)$, which is thus a homogeneous space (for $SO(1,4)$).

We now give a description of the manifold $\mathbb{R} \times S^3$ as a pseudo-sphere (a submanifold) of radius ℓ of the pseudo Euclidean space $\mathbb{R}^{1,4} = \{\mathbb{R}^5, \mathring{\boldsymbol{g}}\}$. If $(X^1, X^2, X^3, X^4, X^0)$ are the

[3] Minkowski spacetime is the structure $\mathfrak{M} = (\mathcal{M} = \mathbb{R}^4, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$ where $\boldsymbol{\eta}$ is the usual Minkowski metric, $\tau_{\boldsymbol{\eta}} \in \sec \bigwedge^4 T^* \mathcal{M}$ defines an orientation and $\uparrow$ denotes that $(\mathcal{M}, \boldsymbol{\eta})$ is time orientable. Details in [3].

[4] It is a vacuum solution of Einstein equation with a cosmological constant term. We are not going to use this structure in this paper.

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global orthogonal coordinates of $\mathbb{R}^{1,4}$, then the equation representing the pseudo sphere is

$$(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 - (X^4)^2 = -\ell^2. \tag{2}$$

Introducing projective *conformal* coordinates $\{x^\mu\}$ by projecting the points of $\mathbb{R}\times S^3$ from the "north-pole" to a plane tangent to the "south pole" we see immediately that $\{x^\mu\}$ covers all $\mathbb{R}\times S^3$ except the "north-pole". We have [2, 5, 7, 8]

$$X^\mu = \Omega x^\mu, \quad X^4 = -\ell\Omega\left(1 + \frac{\sigma^2}{4\ell^2}\right) \tag{3}$$

with

$$\Omega = \left(1 - \frac{\sigma^2}{4\ell^2}\right)^{-1}, \quad \sigma^2 = \eta_{\mu\nu}x^\mu x^\nu \tag{4}$$

and we immediately find that

$$\boldsymbol{g} := \boldsymbol{\iota}^* \mathring{\boldsymbol{g}} = \Omega^2 \eta_{\mu\nu} dx^\mu \otimes dx^\nu, \tag{5}$$

and the matrix with entries $\eta_{\mu\nu}$ is the diagonal matrix $\mathrm{diag}(1, -1, -1, -1)$.

3 Constant Bulk Angular Momentum versus Geodesic Equation

Now, write $\boldsymbol{D}_{\partial_\mu}\partial_\nu = \Gamma^{\alpha\cdot\cdot}_{\cdot\mu\nu}\partial_\alpha$ and let $\sigma : I \to M, s \mapsto \sigma(s)$ be a time like geodesic in M. Its tangent vector field σ_* such that $\sigma_*(s) = \frac{d\boldsymbol{x}^\mu \circ \sigma(s)}{ds} \frac{\partial}{\partial x^\mu}\Big|_\sigma = \frac{dx^\mu}{ds}\frac{\partial}{\partial x^\mu}$ satisfy $\boldsymbol{D}_{\sigma_*}\sigma_* = 0$ and in components it is

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{6}$$

In [6] we obtain the following equation for this geodesic in the de Sitter manifold:

$$\frac{d^2x^\alpha}{ds^2} + \frac{\Omega}{l^2}x_\mu \frac{dx^\mu}{ds}\frac{dx^0}{ds} - \frac{\Omega}{2l^2}x^0\frac{dx_\mu}{ds}\frac{dx^\mu}{ds} = 0. \tag{7}$$

Let $\{\mathbf{E}_A = \frac{\partial}{\partial X^A}\}$, $A = 0, 1, 2, 3, 4$ be the canonical basis of $T\mathring{M} = T\mathbb{R}^5$ and let $\{E^A = dX^A\}$ be a basis of $T^*\mathring{M}$ dual to $\{\mathbf{E}_A = \frac{\partial}{\partial X^A}\}$. We have

$$\mathring{\boldsymbol{g}} = \boldsymbol{\eta}_{AB}E^A \otimes E^B \tag{8}$$

where the matrix with entries $\boldsymbol{\eta}_{AB}$ is the diagonal matrix $\mathrm{diag}(1, -1, -1, -1, -1)$. Moreover let $\mathring{\mathtt{g}} = \eta^{AB}\mathbf{E}_A \otimes \mathbf{E}_B$ be the metric of the cotangent bundle (with $\eta^{AC}\boldsymbol{\eta}_{CB} = \delta^A_B$). Finally let $\{E_A\}$ be the reciprocal basis of $\{E^A\}$, i.e., $\mathring{\mathtt{g}}(E^A, E_B) = \delta^A_B$. We introduce the basis $\{\mathcal{E}_A\}$ of $\mathbb{R}^5$ and make the usual identification $\mathbf{E}_A(p) \simeq \mathbf{E}_A(p') = \mathfrak{E}_A$, $E_A(p) \simeq E_A(p') = \mathcal{E}_A$ for any $p, p' \in \mathbb{R}^5$.

Let $\boldsymbol{X} = X^A\mathcal{E}_A$ be the position covector, $\boldsymbol{P} = m\dot{X}^B\mathcal{E}_B$ the bulk momentum covector and $\boldsymbol{L} = \boldsymbol{X} \wedge \boldsymbol{P}$ the bulk angular momentum of a particle of mass m in the bulk spacetime

$\mathbb{R}^{1,4}$. If the particle is constrained to move "freely"[5] in the submanifold $\mathbb{R}\times S^3$ a natural hypothesis is that its bulk angular momentum is a constant of motion. Now, $\boldsymbol{L} = \boldsymbol{cte}$ implies immediately

$$\frac{1}{2}(X^A \ddot{X}^B - \ddot{X}^A X^B)\mathcal{E}_A \wedge \mathcal{E}_B = 0. \tag{9}$$

Thus, for $\kappa, \iota = 0, 1, 2, 3$ it is $X^\kappa \ddot{X}^\iota - \ddot{X}^\kappa X^\iota = 0$ and $X^\kappa \ddot{X}^4 - \ddot{X}^\kappa X^4 = 0$, so when we use the conformal coordinates we get [6]:

$$x^k \left(\frac{dx^i}{ds} \frac{1}{\ell^2} \Omega^2 x_i \frac{dx^l}{ds} + \Omega \frac{d^2 x^l}{ds^2} \right) - \left(\frac{dx^i}{ds} \frac{1}{\ell^2} \Omega^2 x_i \frac{dx^k}{ds} + \Omega \frac{d^2 x^k}{ds^2} \right), x^l = 0 \tag{10}$$

$$(2\Omega - 1)\frac{d^2 x^k}{ds^2} + \frac{1}{l^2}\Omega(2\Omega - 1)x_i \frac{dx^i}{ds}\frac{dx^k}{ds} - \frac{1}{2l^4}\Omega^2 x_i x_j x^k \frac{dx^i}{ds}\frac{dx^j}{ds} - \frac{1}{2l^2}\Omega x^k \frac{dx_i}{ds}\frac{dx^i}{ds} - \frac{1}{2l^2}\Omega x_i \frac{d^2 x^i}{ds^2} x^k = 0, \tag{11}$$

which are the equations of motion according to the structure M^{dSL}.

With this notations and hypotesis we have proved in [6] the following proposition:

Proposition 3.1. (a): *If a particle travels with geodesic motion in the structure M^{dSL} then its bulk angular momentum $\boldsymbol{L}$ is constant.* (b): *Also, if a particle of mass m constrained to move in M occurs with constant bulk angular $\boldsymbol{L}$ then its motion for an observer living in the brane M is described by a timelike geodesic in the structure M^{dSL}.*

4 Conclusions

We said in the introduction that the de Sitter structure M^{dSL} has been studied by many authors as a possible natural arena for the motion of particles and fields instead of the Minkowski spacetime structure $\mathfrak{M}$. We discussed these issues in [5]. At least we want to emphasize that recently it has been shown in [7] by using the Clifford and spin-Clifford formalisms [3] that the hypothesis that a particle moving freely in $(M, \boldsymbol{g})$ has constant bulk angular momentum leads naturally to the Dirac equation as found in [1] in the de Sitter structure $(M, \boldsymbol{g})$.

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Referências

[1] P. A. M. Dirac. The Electron Wave Equation in De-Sitter Space. *Ann. Math*, volume 36, pages 657-669, 1935.

[5] From a physical point of view the statement moving 'freely' means that observers living in M cannot detect any force acting on the particle.

[2] F. Gürsey. Introduction to Group Theory. *Relativity, Groups and Topology*, pages 91-161. Gordon and Breach, New York, 1964.

[3] W. A. Rodrigues Jr. and E. Capelas de Oliveira. *The Many Faces of Maxwell, Dirac and Einstein Equation,. A Clifford Bundle Approach*, Lecture Notes in Physics 722. Springer, Heidelberg, 2007.

[4] W. A. Rodrigues Jr. and S. Wainer. A Clifford Bundle Approach to the Geometry of Branes. *Adv. Appl. Clifford Algebras*, volume 24, pages 817-847, 2014.

[5] W. A. Rodrigues Jr. and S. Wainer. Notes on Conservation Laws, Equations of Motion of Matter and Particle Fields in Lorentzian and Teleparallel de Sitter Spacetime Structures. 2015, to appear in *Adv. Math. Phys.*

[6] W. A. Rodrigues Jr. and S. Wainer. On the Motion of a Free Particle in the de Sitter Manifold. 2016. *[arXiv:1601.05751 [math-ph]]*

[7] W. A. Rodrigues Jr., S. Wainer, M. Rivera-Tapia, E. A. Notte-Cuello and I. Kondrashuk. A Clifford Bundle Approach to the Wave Equation of a Spin 1/2 Fermion in the de Sitter Manifold. *Adv. Appl. Clifford Algebras*, volume 26, pages 253-277, 2016.

[8] H-J. Schmidt. On the de Sitter Space-Time-The Geometric Foundation of Inflationary Cosmology. *Fortschr. Phys.*, volume 41, pages 179-199, 1933.

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