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Robustness on the class of fuzzy difference operators

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Abstract. This paper extends the robustness analysis of the fuzzy connectives based on the pointwise sensitivity of such operators. Starting with an evaluation of the sensitivity in fuzzy negations, triangular norms and conorms, we apply the results in the class of fuzzy difference operators and their dual construction. The paper formally states that the robustness preserves the projection functions related to fuzzy (co)difference operators.

Keywords. Fuzzy logic, δ sensitivity, robustness, fuzzy (co)difference operators

1 Introduction

Scientific research areas in which the degrees of certainty are only approximately defined, it seems reasonable to require that the corresponding system is the least sensitive to small changes in the inputs. Robustness analysis provides an estimation of perturbation caused by input parameters contributing to identify criteria for choices of fuzzy reasoning methods in real applications as industrial robotics and electronics industrial systems [7–9].

The concepts of maximum and average perturbations of fuzzy sets have been proposed, estimating maximum and average perturbation parameters for various fuzzy reasoning methodologies [3, 10, 11]. In [6, 15], by reducing sensitivity in the corresponding pointwise components of fuzzy connectives it is possible to estimate the sensitivity of fuzzy connectives in output data by propagation from small input changes. Following this approach, this paper consolidates the study of δ-sensitivity of fuzzy connectives (FCs) according with results previously stated in [12,18,19] and related to fuzzy negations, triangular (co)norms, fuzzy (co)implications and fuzzy X(N)or connectives.

Fuzzy difference operators are frequently applied in fuzzy relations in order to obtain similarity, correlation, distance and entropy of fuzzy measures. In particular, this

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paper aims to study the robustness analysis defined on δ-sensitivity of fuzzy difference operators which are characterized as representable fuzzy connectives (FCs), meaning that they can be defined as compositions of aggregations and fuzzy complement. Moreover, as binary connectives, the δ-sensitivity analysis of fuzzy difference operators considers the monotonicity property of both arguments restricted to their left and right projections.

Since δ-sensitivity on interpretation of FCs is closely related to (non-)truth conditional fuzzy rules, this work also focusses on their corresponding class of representable (co)difference operators, according with robustness analysis proposed in [12].

Thus, the δ-sensitivity of fuzzy (co)difference operators is discussed, based on the study of δ-sensitivity of both classes of triangular (co)norms and strong fuzzy negations. When the standard fuzzy negation N_S is considered such analysis is applied to N_S-dual aggregation subclasses, the product triangular norm T_P and the algebraic sum S_P.

The notion of δ-sensitivity on a dual fuzzy connective is relevant to study its fuzzy intuitionistic extension, which is characterized by the non-complementary relationship between the membership and non-membership functions [1]. Such interpretation improves the study of the stability of systems based on intuitionistic fuzzy rules and measurement relationships. In [13], focusing on their pointwise components obtained by related membership and non-membership functions, the δ-sensitivity analysis related to difference operators is extended to the Atanassov's intuitionistic approach [1], as presented in [17]. And, by taking the class of strong fuzzy negation (as the standard negation N_S), the paper formally states that the sensitivity of an n-order fuzzy connective at a point $\mathbf{x} \in U^n$ preserves its projections related to the sensitivity of its fuzzy approach at the same point, when representable fuzzy connectives are considered.

In order to study the robustness on the class of fuzzy (co)difference operators, the paper is organized as follows. Firstly, the preliminaries describe the basic concepts of FCs. The δ-sensitivity of FCs and general results of robustness of FCs are reported in Section 3. Additionally, Subsection 3.1 considers the robustness in the class of fuzzy difference operators, which can be obtained by fuzzy negations and triangular norms, also including their dual construction. Final remarks are considered in the conclusion.

2 Preliminaries

By recalling some basic concepts of FCs we firstly report notions of FL as conceived by Zadeh [16] concerning negations, difference [5] and (co)difference.

Let $U = [0, 1]$ be the unit interval of real numbers. Recall that a function $N : U \to U$ is a **fuzzy negation** if it satisfies, for all$\in U$ the properties:
N1 : $N(0)=1$ and $N(1)=0$; **N2** : If $x \geq y$ then $N(x) \leq N(y)$.
A fuzzy negation satisfying the involutive property:
N3 $N(N(x)) = x$, $\forall x \in U$,
is called a **strong fuzzy negation** (SFN), e.g. the standard negation $N_S(x) = 1 - x$.

When $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in U^n$ and N is a fuzzy negation, the following notation is considered: $N(\mathbf{x}) = (N(x_1), N(x_2), \ldots, N(x_n))$.

Let N be a negation. The N**-dual function** of $f : U^n \to U$ is given by:

$$f_N(\mathbf{x}) = N(f(N(\mathbf{x}))),\ \forall \mathbf{x} \in U^n. \tag{1}$$

A function $T{:}U^2{\to}U$ is a **triangular-norm** (t-norm) if and only if it satisfies, for all $x, y, z \in U$, the following properties.

T1: $T(x, 1) = x$; **S1**: $S(x, 0) = x$;
T2: $T(x, y) = T(y, x)$; **S2**: $S(x, y) = S(y, x)$;
T3: $T(x, T(y, z)) = T(T(x, y), z)$; **S3**: $S(x, S(y, z)) = S(S(x, y), z)$;
T4: if $x \leq x'$ and $y \leq y'$, $T(x, y) \leq T(x', y')$. **S4**: if $x \leq x'$ and $y \leq y'$, $S(x, y) \leq S(x', y')$.

Let N be a fuzzy negation on U. The mappings $T_N, S_N : U^2 \to U$ denote the N-dual functions of a t-norm T and a t-conorm S, respectively defined as:

$$T_N(x, y) = N(T(N(x), N(y))), \qquad S_N(x, y) = N(S(N(x), N(y))). \tag{2}$$

Typical examples of t-norms and t-conorms, respectively, are the following:

1. Minimum and maximum: $T_M(x, y) = \min\{x, y\}$ and $S_M(x, y) = \max\{x, y\}$;

2. Product and probabilistic sum: $T_P(x, y) = xy$ and $S_P(x, y) = x + y - xy$;

3. Łukaziewski t-norm and t-conorm: $T_L(x, x) = max(x + y - 1, 0)$ and $S_L(x, y) = min(x + y, 1)$;

The following definition of fuzzy difference operators extends the axioms from [5, Definition 4] to dual construction of fuzzy co-difference operators.

Definition 2.1. *A function $D(E) : U^2 \to U$ is a* ***fuzzy (co)difference*** *if it satisfies, for all $x, y, z \in U$, the following properties:*

D0*: $D(x, y) \leq x$;* ***E0****: $E(x, y) \geq x$;*
D1*: $D(x, 0) = x$;* ***E1****: $E(x, 1) = x$;*
D2*: $y{\leq}z \to D(x, y){\geq}D(x, z)$;* ***E2****: $y{\leq}z \to E(x, y){\geq}E(x, z)$;*
D3*: $y{\leq}z \to D(y, x){\leq}D(z, x)$;* ***E3****: $y{\leq}z \to E(y, x){\leq}E(z, x)$;*
D4*: $D(1, x) = N_D(x)$ is a fuzzy negation.* *E**3**: $E(0, x) = N_E(x)$ is a fuzzy negation.*

Let N be a fuzzy negation on U. The mappings $D_N, E_N : U^2 \to U$ denote the N-dual functions of a fuzzy (co)difference D (E), respectively defined as:

$$D_N(x, y) = N(D(N(x), N(y))), \qquad E_N(x, y) = N(E(N(x), N(y))). \tag{3}$$

Proposition 2.1. *A fuzzy (co)difference $D_{T,N}, (E_{S,N}) : U^2 \to U$ verifies the properties:*

$$D_{T,N}(x, y) = T(x, N(y)) \qquad E_{S,N}(x, y) = S(N(x), y) \tag{4}$$

Proof. For all $x, y, z \in U$, the following is verified:
D0: If $y \leq 1$ $D_{T,N}(x, y) = T(x, N(y)) \leq T(x, N(0)) = T(x, 1) = x$.
D1: $D(x, 0) = T(x, N(0)) = T(x, 1) = x$.
D2: If $x \leq y$ then $D(x, z) = T(x, N(z)) \leq T(y, N(z)) = D(y, z)$.

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Table 1: Examples in the classes $\mathcal{C}(D_{T,N})$ and $\mathcal{C}(E_{S,N})$

$D_{T,N}(x,y) = T(x,N(y))$	$E_{S,N}(x,y) = S(x,N(y))$
$D_{T_P,N_S}(x,y) = x - xy$	$E_{S_P,N_S}(x,y) = 1 - y + xy$
$D_{T_M,N_S}(x,y) = \min(x, 1-y)$	$E_{S_M,N_S}(x,y) = \max(x, 1-y)$
$D_{T_L,N_S}(x,y) = \max(x-y, 0)$	$E_{S_L,N_S}(x,y) = \min(x-y+1, 1)$

D3: If $x \leq y$, it implies that $D(z,x) = T(z,N(x)) \geq T(z,N(y)) = D(z,y)$.
D4: $D(1,x) = T(1,N(x)) = N(x)$.
Therefore, $D_{T,N}$ fulfils **D0** $-$ **D4** in Definition2.1. Analogously, one can prove that $E_{S,N}$ fulfils **E0** $-$ **E4**. Therefore, Proposition 2.1 is verified. □

Based on typical examples of t-norms and t-conorms previously reported, Table 1 presents some examples in the classes $\mathcal{C}(D_{T,N})$ and $\mathcal{C}(E_{S,N})$, related to fuzzy difference operators and corresponding N_S-dual constructions.

3 Pointwise sensitivity of fuzzy connectives

Based on [6] and [12], the study of a δ-sensitivity of $n-$order function f at point $\mathbf{x}$ on the domain U is considered, in the context of robustness of fuzzy logic, mainly related to the class of (S,N)-implications.

Definition 3.1. *[6, Definition 1] Let $f : U^n \to U$ be an $n-$order function, $\delta \in U$ and $\mathbf{x} = (x_1, x_2, \ldots x_n)$, $\mathbf{y} = (y_1, y_2, \ldots y_n) \in U^n$. The* ***$\delta$-sensitivity of f at point $\mathbf{x}$****, denoted by $\Delta_f(\mathbf{x}, \delta)$, is given by*

$$\Delta_f(\mathbf{x}, \delta) = \sup\{|f(\mathbf{x}) - f(\mathbf{y})| : \mathbf{y} \in U^n \text{ and } \bigvee(\mathbf{x}, \mathbf{y}) \leq \delta\} \tag{5}$$

wherever $\bigvee(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : i = 1, \ldots, n\}$. Additionally, ***the maximum δ sensitivity of*** *f, denoted as $\Delta_f(\delta)$, is defined as follows:*

$$\Delta_f(\delta) = \bigvee_{\mathbf{x} \in U^n} \Delta_f(\mathbf{x}, \delta). \tag{6}$$

Proposition 3.1. *[12, Theorem 1] If $N = N_S$ and f_N is the N-dual function of f then* ***the sensitivity of f_N at point $\mathbf{x}$*** *is given by*

$$\Delta_{f_N}(\mathbf{x}, \delta) = \Delta_f(N(\mathbf{x}), \delta). \tag{7}$$

Now, we investigate the δ-sensitivity in FCs, in terms of Definition 3.1 based on results previously presented in [6]. In order to provide an easier notation, when $f : U^2 \to U$ and $\mathbf{x} = (x,y) \in U^2$, consider the following notations:

$$f\lfloor \mathbf{x} \rceil \equiv f((x-\delta) \vee 0, (y+\delta) \wedge 1); \qquad f\lceil \mathbf{x} \rfloor \equiv f((x+\delta) \wedge 1, (y-\delta) \vee 0).$$

Table 2: Examples in the classes $\mathcal{C}(D_{T,N})$ and $\mathcal{C}(E_{S,N})$which are related to fuzzy difference operators and their corresponding N_S-dual constructions

	$\Delta_{D_{T_P,N_S}}((0,0),\delta)$	$\Delta_{D_{T_M,N_S}}((0,0),\delta)$	$\Delta_{D_{T_L,N_S}}((0,0),\delta)$
$\mathbf{x}=(0,0)$	δ	δ	δ
$\mathbf{x}=(0,1)$	δ^2	δ	δ
$\mathbf{x}=(1,0)$	δ	δ	2δ
$\mathbf{x}=(1,1)$	$2\delta-\delta^2$	δ	δ

Proposition 3.2. *[6, Theorem 1] Consider $f:U^2\to U$, $\delta\in U$ and $\mathbf{x}=(x,y)\in U^2$. The sensitivity of f at point $\mathbf{x}$ is given by the folllowing cases:*
(i) if f is increasing w.r.t. its variables then we have that:

$$\Delta_f(\mathbf{x},\delta) \;=\; (f\lceil\mathbf{x}\rceil - f_N(\mathbf{x}))\vee(f\lfloor\mathbf{x}\rfloor - f(\mathbf{x})); \tag{8}$$

(ii) if f verifies both properties, 1−place isotonicity and 2−place antitonicity, then:

$$\Delta_f(\mathbf{x},\delta) \;=\; (f\lfloor\mathbf{x}\rceil - f(\mathbf{x}))\vee(f(\mathbf{x}) - f\lceil\mathbf{x}\rfloor) \tag{9}$$

Proposition 3.3. *[12] The δ-sensitivity of a t-(co)norm T (S) is given by Eq. (8):*

$$\Delta_T(\mathbf{x},\delta) \;=\; (T\lfloor\mathbf{x}\rceil - T(\mathbf{x}))\vee(T(\mathbf{x}) - T\lceil\mathbf{x}\rfloor) \tag{10}$$
$$\Delta_S(\mathbf{x},\delta) \;=\; (S\lfloor\mathbf{x}\rceil - S(\mathbf{x}))\vee(S(\mathbf{x}) - S\lceil\mathbf{x}\rfloor) \tag{11}$$

3.1 δ-sensibility of fuzzy (co)difference operators

In this section we study the robustness of fuzzy (co)difference operators based on the δ-sensibility analysis.

Proposition 3.4. *The δ-sensitivity of the fuzzy (co)difference operator is given by Eq. (12):*

$$\Delta_D(\mathbf{x},\delta) \;=\; (D\lfloor\mathbf{x}\rceil - D(\mathbf{x}))\vee(D(\mathbf{x}) - D\lceil\mathbf{x}\rfloor) \tag{12}$$
$$\Delta_E(\mathbf{x},\delta) \;=\; (E\lfloor\mathbf{x}\rceil - E(\mathbf{x}))\vee(E(\mathbf{x}) - E\lceil\mathbf{x}\rfloor) \tag{13}$$

Proof. Straightforward Propositions 2.1. □

Remark 3.1. *By Eqs. (12) and (7) for the first pair of N_S-dual fuzzy difference operators presented in Table 1, we obtain the following:*
$\Delta_{D_{T_P,N_S}}((0,0),\delta)=\delta\vee 0=\delta=\Delta_{E_{S_P,N_S}}((1,1),\delta)$ *;*
$\Delta_{D_{T_P,N_S}}((0,1),\delta)=\delta^2\vee 0=\delta^2=\Delta_{E_{S_P,N_S}}((1,0),\delta)$ *;*
$\Delta_{D_{T_P,N_S}}((1,1),\delta)=\delta\vee 0=\delta=\Delta_{E_{S_P,N_S}}((0,0),\delta)$*; and*
$\Delta_{D_{T_P,N_S}}((1,0),\delta)=0\vee(2\delta-\delta^2)=\Delta_{E_{S_P,N_S}}((0,1),\delta)$.
Analogously, it is easy to obtain the δ sensitivity at the endpoints of U for the other pairs of N_S-dual fuzzy difference operators reported in Table 1. See all results in Table 2.

Proposition 3.5. *The maximum δ sensitivity of a (co)difference operator is given by:*

$$\Delta_D(\delta) = \bigvee_{\mathbf{x}\in U^n} \Delta_D(\mathbf{x},\delta) = \bigvee_{\mathbf{x}\in U^n} \Delta_E(\mathbf{x},\delta) = \Delta_E(\delta) \tag{14}$$

Proof. Straightforward. □

4 Conclusion

The main contribution of this work is concerned with the study of robustness on fuzzy approach related to the fuzzy (co)difference operators which can be obtained by aggregation operators as t-(co)norms and of (strong) fuzzy negations. Additional studies, considering δ-sensitivity of symmetric difference operator and their corresponding dual construction should be carried out. In further work, we focus on the sensitivity of fuzzy inference dependent on intuitionistic fuzzy rules based on intuitionistic fuzzy connectives, including the extension of the robustness studies of R-(co)implications. Furthermore, this project aims to investigate an application of its main results in the robustness analysis of operators (erosion, dilation, closing, opening) used in the mathematical morphology.

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