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On the Sufficiency of the Maximum Principle for Infinite Horizon Optimal Control Problems

Valeriano Antunes de Oliveira[1]

Instituto de Biociências, Letras e Ciências Exatas, UNESP – Univ. Estadual Paulista, Câmpus de São José do Rio Preto, Departamento de Matemática Aplicada

Abstract. This work is devoted to introduce a new sufficient optimality condition for infinite horizon optimal control problems. It is showed that normal extremal processes are optimal under this new condition, termed as MP-pseudoinvexity. Moreover, problems in which every normal extremal process is optimal necessarily obey the definition of MP-pseudoinvexity.

Keywords. Optimal Control, Infinite Horizon, Sufficient Optimality Conditions

1 Introduction

The maximum principle provides necessary optimality conditions for optimal control problems. In the case of infinite horizon problems, it can be found, for example, in Aseev and Kryazhimskiy [1], Tauchnitz [2] and Ye [3]. It is well known in optimal control theory that, in some cases, the necessary conditions of the maximum principle are also sufficient optimality conditions; problems with quadratic cost and linear dynamics and convex problems, for instance. The sufficient optimality conditions furnished here are based in the notion of KT-invexity (see Martin [4]), which is a type of generalized invexity. Invex functions where introduced by Hanson [5] in a study on the sufficiency of the KKT conditions for nonlinear mathematical programming. The notion of KT-invexity was initially studied in the optimal control context in de Oliveira, Silva and Rojas-Medar [6], where KKT type optimality conditions where utilized. In Oliveira, Silva and Rojas-Medar [7], the notion of MP-pseudoinvexity was introduced, when the maximum principle was used. The nonsmooth case was treated in de Oliveira and Silva [8, 9].

This work is devoted to study new sufficient optimality conditions for infinite horizon optimal control problems posed as follows:

$$\begin{aligned} &\text{maximize} && J(x,u)=\int_0^\infty e^{-\delta t}L(x(t),u(t))\,\mathrm{d}t \\ &\text{subject to} && \dot{x}(t)=f(x(t),u(t)) \text{ a.e. } t\in[0,\infty),\ x(0)\in C, \\ &&& x(t)\in E\ \forall\ t\in[0,\infty),\ u(t)\in U \text{ a.e. } t\in[0,\infty), \end{aligned} \tag{P}$$

where $L:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$, $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$, $C\subset\mathbb{R}^n$ is closed, $E\subset\mathbb{R}^n$ is open and $U\subset\mathbb{R}^m$ is Borel measurable. The state variable x is a locally absolutely continuous function from $[0,\infty)$ into $\mathbb{R}^n$ and the control variable u is a measurable function from $[0,\infty)$ into $\mathbb{R}^m$.

[1]antunes@ibilce.unesp.br

We assume, throughout the paper, that f and L are continuously differentiable with respect to the state variable x uniformly in $u \in \mathbb{R}^m$; f and L are Borel measurable with respect to the control variable u; and $L : E \times \mathbb{R}^m \to \mathbb{R}$ is bounded.

We say that (x, u) is an *admissible control process* if $u(t) \in U$ a.e. $t \in [0, \infty)$, x is a trajectory corresponding to the control u such that $x(0) \in C$. We say that an admissible control process $(\bar{x}, \bar{u})$ is an *optimal control process* if $J(x, u) \leq J(\bar{x}, \bar{u})$ for all admissible processes (x, u).

Let $H : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ denote the Pontryagin Hamiltonian function

$$H(t, x, p, u, \lambda) := p \cdot f(x, u) + \lambda e^{-\delta t} L(x, u).$$

Theorem 1.1 (Ye [3]). *If $(\bar{x}, \bar{u})$ is an optimal process of (P), then there exist a scalar λ (equal to 0 or 1) and a locally absolutely continuous function $p : [0, \infty) \to \mathbb{R}^n$ such that*

$$-\dot{p}(t) = f_x(\bar{x}(t), \bar{u}(t))^\top p(t) + \lambda e^{-\delta t} L_x(\bar{x}(t), \bar{u}(t)) \text{ a.e. } t \in [0, \infty), \tag{1}$$

$$\max_{u \in U} H(t, \bar{x}(t), p(t), u, \lambda) = H(t, \bar{x}(t), p(t), \bar{u}(t), \lambda) \tag{2}$$

$$= \lambda\delta \int_t^\infty e^{-\delta s} L(\bar{x}(s), \bar{u}(s))\, \mathrm{d}s \text{ a.e. } t \in [0, \infty), \tag{3}$$

$$p(0) \in N_C(\bar{x}(0)), \ \lim_{t \to \infty} \max_{u \in U} H(t, \bar{x}(t), p(t), u, \lambda) = 0, \tag{4}$$

$$|(p(0), \lambda)| > 0. \tag{5}$$

When there exist λ and p satisfying (1)-(5) we say that $(\bar{x}, \bar{u})$ is an *extremal process* of (P). If $\lambda = 1$, we say that $(\bar{x}, \bar{u})$ is a normal extremal process and (P) is said to be *normal* at $(\bar{x}, \bar{u})$. We say that (P) is *normal* if it is normal at any extremal process $(\bar{x}, \bar{u})$.

The assumption (A) is said to be valid at an admissible process $(\bar{x}, \bar{u})$ if L is nonnegative and $0 \in \operatorname{int}\{f(\bar{x}(t), u) : u \in U\}$ for all t large.

Corollary 1.1 (Ye [3]). *Let $(\bar{x}, \bar{u})$ be an optimal process of (P). Assume that assumption (A) is valid at $(\bar{x}, \bar{u})$. Then, in addition to conditions (1)-(5), the transversality condition*

$$\lim_{t \to \infty} p(t) = 0 \tag{6}$$

is also verified.

2 MP-pseudoinvexity and Optimality Conditions

Given $(t, x, p, u, \lambda) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$, the map $\Delta H(t, x, p, u, \lambda) : \mathbb{R}^m \to \mathbb{R}$ is defined as $\Delta H(t, x, p, u, \lambda)(v) := H(t, x, p, v, \lambda) - H(t, x, p, u, \lambda)$. Similarly, $\Delta f(x, u)(v) := f(x, v) - f(x, u)$ and $\Delta L(x, u)(v) := L(x, v) - L(x, u)$.

Let $(\bar{x}, \bar{u})$ be a admissible process. A triple $(y, z, v) : [0, \infty) \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$ is said to be an *admissible variation* along $(\bar{x}, \bar{u})$ if y and z are bounded locally absolutely continuous functions, v is measurable, and

$$\dot{y}(t) = f_x(\bar{x}(t), \bar{u}(t))y(t) + \Delta f(\bar{x}(t), \bar{u}(t))(v(t)) + f(\bar{x}(t), v(t))\dot{z}(t) \text{ a.e. } t \in [0, \infty),$$
$$y(0) \in T_C(\bar{x}(0)), \ z(0) \geq 0, \ v(t) \in U \text{ a.e. } t \in [0, \infty).$$

The set of all admissible variations along $(\bar{x}, \bar{u})$ will be denoted as $\mathcal{V}(\bar{x}, \bar{u})$.

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Definition 2.1. *Let* $(\bar{x}, \bar{u})$ *be an admissible process. (P) is said to be* MP-pseudoinvex *at* $(\bar{x}, \bar{u})$ *if for each admissible process* (x, u)*, there exists* $(\eta, \nu, \xi) \in \mathcal{V}(\bar{x}, \bar{u})$ *such that*

$$J(x,u) - J(\bar{x},\bar{u}) > 0 \;\Rightarrow\; \int_0^\infty e^{-\delta t}[L_x(\bar{x}(t),\bar{u}(t)) \cdot \eta(t) + \Delta L(\bar{x}(t),\bar{u}(t))(\xi(t))]\,\mathrm{d}t + \int_0^\infty e^{-\delta t}[L(\bar{x}(t),\xi(t))\dot{\nu}(t) - \delta L(\bar{x}(t),\bar{u}(t))\nu(t)]\,\mathrm{d}t > 0. \quad (7)$$

(P) is said to be MP-pseudoinvex *if it is MP-pseudoinvex at every admissible process.*

Theorem 2.1. *Let* $(\bar{x}, \bar{u})$ *be a normal extremal process. Assume that (P) is MP-pseudoinvex at* $(\bar{x}, \bar{u})$ *and that assumption (A) holds at* $(\bar{x}, \bar{u})$*. Then* $(\bar{x}, \bar{u})$ *is an optimal process.*

Proof. Provided $(\bar{x}, \bar{u})$ is a normal extremal process and assumption (A) holds, conditions (1)-(6) hold with $\lambda = 1$. If it is not optimal, there exists an admissible process (x, u) such that $J(x, u) > J(\bar{x}, \bar{u})$. Provided (P) is MP-pseudoinvex, there exist $(\eta, \nu, \xi) \in \mathcal{V}(\bar{x}, \bar{u})$ so that (7) is valid.

Set $q(t) = -\delta \int_t^\infty e^{-\delta s} L(\bar{x}(s), \bar{u}(s))\,\mathrm{d}s$, $t \in [0, \infty)$, so that $\dot{q}(t) = \delta e^{-\delta t} L(\bar{x}(t), \bar{u}(t))$ a.e. $t \in [0, \infty)$ and $q(t) \to 0$, $t \to \infty$. By (3), $H(t, \bar{x}(t), p(t), \bar{u}(t), \lambda) = -q(t)$ a.e. $t \in [0, \infty)$.

It follows that

$$\begin{aligned}
0 \;&<\; \int_0^\infty p(t) \cdot [f_x(\bar{x}(t),\bar{u}(t))\eta(t) + \Delta f(\bar{x}(t),\bar{u}(t))(\xi(t)) + f(\bar{x}(t),\xi(t))\dot{\nu}(t)]\,\mathrm{d}t \\
&\quad - \int_0^\infty p(t)\cdot\dot{\eta}(t)\,\mathrm{d}t + \int_0^\infty e^{-\delta t}[L_x(\bar{x}(t),\bar{u}(t))\cdot\eta(t) + \Delta L(\bar{x}(t),\bar{u}(t))(\xi(t))]\,\mathrm{d}t \\
&\quad + \int_0^\infty e^{-\delta t}[L(\bar{x}(t),\xi(t))\dot{\nu}(t) - \delta L(\bar{x}(t),\bar{u}(t))\nu(t)]\,\mathrm{d}t \\
&= \int_0^\infty [f_x(\bar{x}(t),\bar{u}(t))^\top p(t) + e^{-\delta t} L_x(\bar{x}(t),\bar{u}(t))]\cdot\eta(t)\,\mathrm{d}t \\
&\quad - \lim_{t\to\infty} p(t)\cdot\eta(t) + p(0)\cdot\eta(0) + \int_0^\infty \dot{p}(t)\cdot\eta(t)\,\mathrm{d}t \\
&\quad + \int_0^\infty [p(t)\cdot\Delta f(\bar{x}(t),\bar{u}(t))(\xi(t)) + e^{-\delta t}\Delta L(\bar{x}(t),\bar{u}(t))(\xi(t))]\,\mathrm{d}t \\
&\quad + \int_0^\infty [p(t)\cdot f(\bar{x}(t),\xi(t)) + e^{-\delta t} L(\bar{x}(t),\xi(t))]\dot{\nu}(t)\,\mathrm{d}t \\
&\quad - \int_0^\infty \delta e^{-\delta t} L(\bar{x}(t),\bar{u}(t))\nu(t)\,\mathrm{d}t \\
&= p(0)\cdot\eta(0) + \int_0^\infty \Delta H(t,\bar{x}(t),p(t),\bar{u}(t),1)(\xi(t))\,\mathrm{d}t \\
&\quad + \int_0^\infty H(t,\bar{x}(t),p(t),\xi(t),1)\dot{\nu}(t)\,\mathrm{d}t - \int_0^\infty \dot{q}(t)\nu(t)\,\mathrm{d}t \\
&= p(0)\cdot\eta(0) + \int_0^\infty \Delta H(t,\bar{x}(t),p(t),\bar{u}(t),1)(\xi(t))\,\mathrm{d}t \\
&\quad + \int_0^\infty H(t,\bar{x}(t),p(t),\xi(t),1)\dot{\nu}(t)\,\mathrm{d}t - \lim_{t\to\infty} q(t)\nu(t) + q(0)\nu(0) + \int_0^\infty q(t)\dot{\nu}(t)\,\mathrm{d}t
\end{aligned}$$

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$$
\begin{aligned}
= \quad & p(0)\cdot\eta(0) + \int_0^\infty \Delta H(t,\bar{x}(t),p(t),\bar{u}(t),1)(\xi(t))\,\mathrm{d}t \\
& + q(0)\nu(0) + \int_0^\infty [H(t,\bar{x}(t),p(t),\xi(t),1) + q(t)]\dot{\nu}(t)\,\mathrm{d}t \le 0,
\end{aligned}
$$

a contradiction. Thus $(\bar{x},\bar{u})$ is an optimal process. □

The following result is an immediate consequence of the last theorem.

Theorem 2.2. *Suppose that assumption (A) is valid for every extremal process. If (P) is MP-pseudoinvex, then every normal extremal process is an optimal process.*

Proposition 2.1. *Let $(\bar{x},\bar{u})$ be an admissible process of (P). Assume that for every admissible process (x,u) such that $J(x,u) > J(\bar{x},\bar{u})$, there exist $\eta_0 \in T_C(\bar{x}(0))$, a locally absolutely continuous function $\nu : [0,\infty) \to \mathbb{R}$ and a measurable function $\xi : [0,\infty) \to \mathbb{R}^m$ satisfying*

$$
\xi(t) \in U \text{ a.e. } t \in [0,\infty), \;\; \nu(0) \ge 0, \tag{8}
$$

$$
\begin{aligned}
p(0)\cdot\eta_0 &+ \int_0^\infty \Delta H(t,\bar{x}(t),p(t),\bar{u}(t),1)(\xi(t))\,\mathrm{d}t \\
&+ q(0)\nu(0) + \int_0^\infty [H(t,\bar{x}(t),p(t),\xi(t),1) + q(t)]\dot{\nu}(t)\,\mathrm{d}t > 0
\end{aligned} \tag{9}
$$

for all locally absolutely continuous function $p : [0,\infty) \to \mathbb{R}^n$ verifying

$$
-\dot{p}(t) = f_x(\bar{x}(t),\bar{u}(t))^\top p(t) + e^{-\delta t} L_x(\bar{x}(t),\bar{u}(t)) \text{ a.e. } t \in [0,\infty), \tag{10}
$$

$$
p(0) \in N_C(\bar{x}(0)), \;\; \lim_{t\to\infty} p(t) = 0, \tag{11}
$$

where $q(t) = -\delta \int_t^\infty e^{-\delta s} L(\bar{x}(s),\bar{u}(s))\,\mathrm{d}s$, $t \in [0,\infty)$. Then it is MP-pseudoinvex at $(\bar{x},\bar{u})$.

Proof. Let (x,u) be an admissible process of (P) such that $J(x,u) > J(\bar{x},\bar{u})$ and $p(t)$ satisfying (10)-(11). Then, by hypothesis, there exist a vector $\eta_0 \in T_C(\bar{x}(0))$ and functions ν and ξ verifying (8) and (9). Let $\eta(t)$ be a solution of

$$
\begin{aligned}
&\dot{\eta}(t) = f_x(\bar{x}(t),\bar{u}(t))\eta(t) + f(\bar{x}(t),\xi(t))\dot{\nu}(t) + \Delta f(\bar{x}(t),\bar{u}(t))(\xi(t)) \text{ a.e. } t \in [0,\infty), \\
&\eta(0) = \eta_0.
\end{aligned}
$$

Thence $(\eta,\nu,\xi) \in \mathcal{V}(\bar{x}(t),\bar{u}(t))$. Proceeding as in the proof of Theorem 2.1, we get

$$
\begin{aligned}
&\int_0^\infty e^{-\delta t}[L_x(\bar{x}(t),\bar{u}(t))\cdot\eta(t) + \Delta L(\bar{x}(t),\bar{u}(t))(\xi(t))]\,\mathrm{d}t \\
&\quad + \int_0^\infty e^{-\delta t}[L(\bar{x}(t),\xi(t))\dot{\nu}(t) - L(\bar{x}(t),\bar{u}(t))\nu(t)]\,\mathrm{d}t
\end{aligned}
$$

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$$\begin{aligned}
&= \int_0^\infty p(t)\cdot[f_x(\bar{x}(t),\bar{u}(t))\eta(t) + \Delta f(\bar{x}(t),\bar{u}(t))(\xi(t)) + f(\bar{x}(t),\xi(t))\dot{\nu}(t)]\,\mathrm{d}t\\
&\quad - \int_0^\infty p(t)\cdot\dot{\eta}(t)\,\mathrm{d}t + \int_0^\infty e^{-\delta t}[L_x(\bar{x}(t),\bar{u}(t))\cdot\eta(t) + \Delta L(\bar{x}(t),\bar{u}(t))(\xi(t))]\,\mathrm{d}t\\
&\quad + \int_0^\infty e^{-\delta t}[L(\bar{x}(t),\xi(t))\dot{\nu}(t) - L(\bar{x}(t),\bar{u}(t))\nu(t)]\,\mathrm{d}t\\
&= p(0)\cdot\eta(0) + \int_0^\infty \Delta H(t,\bar{x}(t),p(t),\bar{u}(t),1)(\xi(t))\,\mathrm{d}t\\
&\quad + q(0)\nu(0) + \int_0^\infty [H(t,\bar{x}(t),p(t),\xi(t),1) + q(t)]\dot{\nu}(t)\,\mathrm{d}t.
\end{aligned}$$

So, from (9) we see that (7) is also satisfied. Therefore (P) is MP-pseudoinvex at $(\bar{x},\bar{u})$. □

Proposition 2.2. *Let $(\bar{x},\bar{u})$ be an admissible process of (P). Assume that given p satisfying (10)-(11), inequality (9) does not hold for all $\eta_0 \in T_C(x^*(0))$ and $(\nu,\xi):[0,\infty)\to\mathbb{R}\times\mathbb{R}^m$ satisfying (8). Then $(\bar{x},\bar{u})$ is a normal extremal process of (P).*

Proof. It is enough to show that (2)-(3) are verified with $\lambda = 1$. Define $\widetilde{H}:[0,\infty)\times\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}$ as $\widetilde{H}(t,u,v) := v[H(t,\bar{x}(t),p(t),u,1)+q(t)]$, $q(t) = -\delta\int_t^\infty e^{-\delta s}L(\bar{x}(s),\bar{u}(s))\,\mathrm{d}s$, $t\in[0,\infty)$, and $\widetilde{U} := U\times(1/2,\infty)$. We claim that $\max\{\widetilde{H}(t,u,v) : (u,v)\in\widetilde{U}\} = \widetilde{H}(t,\bar{u}(t),\bar{v}(t))$ a.e. $t\in[0,\infty)$, for $\bar{v}\equiv 1$. If it is not so, there exists $\epsilon>0$ such that

$$\sup_{(u,v)\in\widetilde{U}} \widetilde{H}(t,u,v) - \epsilon > \widetilde{H}(t,\bar{u}(t),\bar{v}(t)) \;\forall\, t\in A, \tag{12}$$

where $A\subset[0,\infty)$ has positive measure. Consider the multifunction $U':[0,\infty)\rightrightarrows\mathbb{R}^m\times\mathbb{R}$ defined as $U'(t) := \{(u,v)\in\widetilde{U} : \phi(t,u,v)>0\}$, where

$$\phi(t,u,v) = \begin{cases} \widetilde{H}(t,u,v) - \widetilde{H}(t,\bar{u}(t),\bar{v}(t)), & \text{for } t\in A,\\ 1, & \text{for } t\in[0,\infty)\setminus A.\end{cases}$$

From (12) we see that U' is a nonempty multifunction. Also, $\mathrm{Gr}(U') = \phi^{-1}((0,\infty))\cap[0,\infty)\times\widetilde{U}$, from where we see that $\mathrm{Gr}(U')$ is $\mathcal{L}\times\mathcal{B}^m\times\mathcal{B}$ measurable. It follows from the Aumann's Measurable Selection Theorem that U' has a measurable selection, which means that there exists a measurable function $(\tilde{u},\tilde{v})$ such that $(\tilde{u},\tilde{v})(t)\in\widetilde{U}$ a.e. $t\in[0,\infty)$ and $\widetilde{H}(t,\tilde{u}(t),\tilde{v}(t)) > \widetilde{H}(t,\bar{u}(t),\bar{v}(t))$ a.e. $t\in A$. Thence

$$\int_A[\widetilde{H}(t,\tilde{u}(t),\tilde{v}(t)) - \widetilde{H}(t,\bar{u}(t),\bar{v}(t))]\,\mathrm{d}t > 0. \tag{13}$$

Let $\eta = 0\in T_C(\bar{x}(0))$, and take

$$\xi(t) = \begin{cases}\tilde{u}(t), & \text{for } t\in A,\\ \bar{u}(t), & \text{for } t\in[0,\infty)\setminus A,\end{cases} \qquad \rho(t) = \begin{cases}\tilde{v}(t)-1, & \text{for } t\in A,\\ 0, & \text{for } t\in[0,\infty)\setminus A,\end{cases}$$

and $\nu(t) = \nu_0 + \int_0^t \rho(s)\,\mathrm{d}s$, $t \in [0,\infty)$, where $\nu(0) = \nu_0 = 0$. Then (8) holds true. By hypothesis we have that

$$\begin{aligned} p(0)\cdot\eta_0 \quad &+ \quad \int_0^\infty \Delta H(t,\bar{x}(t),p(t),\bar{u}(t),\lambda)(\xi(t))\,\mathrm{d}t \\ &+ \quad q(0)\nu(0) + \int_0^\infty [H(t,\bar{x}(t),p(t),\xi,1) + q(t)]\dot{\nu}(t)\,\mathrm{d}t \leq 0. \end{aligned}$$

But then

$$\begin{aligned} 0 \quad &\geq \quad \int_0^\infty \Delta H(t,\bar{x}(t),p(t),\bar{u}(t),\lambda)(\xi(t))\,\mathrm{d}t + \int_0^\infty [H(t,\bar{x}(t),p(t),\xi(t),1) + q(t)]\dot{\nu}(t)\,\mathrm{d}t \\ &= \quad \int_A [H(t,\bar{x}(t),p(t),\tilde{u}(t),1) - H(t,\bar{x}(t),p(t),\bar{u}(t),1)]\,\mathrm{d}t \\ &\quad\; + \int_A [H(t,\bar{x}(t),p(t),\tilde{u}(t),1) + q(t)][\tilde{v}(t) - 1]\,\mathrm{d}t \\ &= \quad \int_A \{\tilde{v}(t)[H(t,\bar{x}(t),p(t),\tilde{u}(t),1) + q(t)] - [H(t,\bar{x}(t),p(t),\bar{u}(t),1) + q(t)]\}\,\mathrm{d}t \\ &= \quad \int_A [\widetilde{H}(t,\tilde{u}(t),\tilde{v}(t)) - \widetilde{H}(t,\bar{u}(t),\bar{v}(t))]\,\mathrm{d}t, \end{aligned}$$

which is in disagreement to (13). Therefore,

$$\widetilde{H}(t,u,v) \leq \widetilde{H}(t,\bar{u}(t),\bar{v}(t)) \;\forall\; (u,v) \in \widetilde{U} \text{ a.e. } t \in [0,\infty).$$

For $v \equiv 1$, we obtain $\widetilde{H}(t,u,1) \leq \widetilde{H}(t,\bar{u}(t),1) \;\forall\; u \in U$ a.e. $t \in [0,\infty)$, or,

$$H(t,\bar{x}(t),p(t),u,1) \leq H(t,\bar{x}(t),p(t),\bar{u}(t),1) \;\forall\; u \in U \text{ a.e. } t \in [0,\infty),$$

so that (2) holds. For almost every $t \in [0,\infty)$, putting $u = \bar{u}(t)$, we have

$$\widetilde{H}(t,\bar{u}(t),v) \leq \widetilde{H}(t,\bar{u}(t),\bar{v}(t)) \;\forall\; v \in (1/2,\infty) \text{ a.e. } t \in [0,\infty).$$

Then we shall have $\widetilde{H}_v(t,\bar{u}(t),\bar{v}(t)) = 0$ a.e. $t \in [0,\infty)$, that is,

$$H(t,\bar{x}(t),p(t),\bar{u}(t),\lambda) + q(t) = 0 \text{ a.e. } t \in [0,\infty),$$

so that (3) is verified. Thus $(\bar{x},\bar{u})$ is a normal extremal process, as aimed. □

Theorem 2.3. *Assume that every normal extremal process is an optimal process. Then (P) is MP-pseudoinvex.*

Proof. Let $(\bar{x},\bar{u})$ be an admissible process and suppose that (P) is not MP-pseudoinvex at $(\bar{x},\bar{u})$. Then, by Proposition 2.1, there exists an admissible process (x,u) such that $J(x,u) > J(\bar{x},\bar{u})$ and an p satisfying (10)-(11) such that (9) does not hold for all $\eta_0 \in T_C(x^*(0))$ and $(\nu,\xi) : [0,\infty) \to \mathbb{R}\times\mathbb{R}^m$ satisfying (8). Hence, by Proposition 2.2, $(\bar{x},\bar{u})$ is a normal extremal process, and then, by hypothesis, an optimal process. This contradicts the fact that $J(x,u) > J(\bar{x},\bar{u})$. Thus (P) is MP-pseudoinvex. □

The theorem below follows directly from Theorems 2.2 and 2.3.

Theorem 2.4. *Assume that assumption (A) is valid at each extremal process. Then (P) is MP-pseudoinvex if, and only if, every extremal process is an optimal process.*

3 Conclusions

The notion of MP-pseudoinvexity was generalized for the context of infinite horizon optimal control problems. It was showed (see Theorems 2.1 and 2.2) that for MP-pseudoinvex problems, the necessary optimality conditions of the maximum principle becomes also sufficient. Moreover, it was established (see Theorems 2.3 and 2.4) that the class of MP-pseudoinvex problems constitutes the most general class of problems where this important property is verified.

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