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Critical set of the Kawahara equation

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Abstract. We characterize the lengths of intervals for which the linear Kawahara equation has a non-trivial solution, whose energy is stationary. This gives rise to a family of complex functions. Characterizing the lengths amounts to deciding which members of this family are entire functions. Our approach is essentially based on determining the existence of certain Möbius transformation.

Key-words. Entire functions, Möbius transformations, stationary solutions, Kawahara equation

1 Introduction

In the Kawahara equation $u_t + u_x + \kappa u_{xxx} + \eta u_{xxxxx} + uu_x = 0$, the conservative dispersive effect is represented by the term $(\kappa u_{xxx} + \eta u_{xxxxx})$. This equation is a model for plasma wave, capilarity-gravity water waves and other dispersive phenomena when the cubic KdV-type equation is weak. Kawahara [6] pointed out that it happens when the coefficient of the third order derivative in the KdV equation becomes very small or even zero. It is then necessary to take into account the higher order effect of dispersion in order to balance the nonlinear effect.

Dispersive problems have been object of intensive research (see, for instance, the classical paper of Benjamin, Bona and Mahoni [2], Biagioni and Linares [3], Bona and Chen [4], Menzala *et al.* [8], Rosier [9], and references therein). Recently global stabilization of the generalized KdV system have been obtained by Rosier and Zhang [10] and Linares and Pazoto [7] studied the stabilization of the generalized KdV system with critical exponents. For the stabilization of global solutions of the Kawahara under the effect of a localized damping mechanism, see Vasconcellos and Silva [11–13].

We consider the linear Kawahara equation

$$u_t + \beta u_x + \kappa u_{xxx} + \eta u_{xxxxx} = 0 \quad \text{with } (x,t) \in (0, L_0) \times (0, \infty), \tag{1}$$

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where the coefficients β, κ and η are real numbers such that $\eta < 0$, $\kappa \neq 0$, $\beta \in \{0, 1\}$. Sometimes, while discussing the existence of solutions of certain partial differential equations, it is necessary to establish when a certain quotient of entire functions still turns out to be an entire function (see, for instance, Rosier [9], Vasconcellos and Silva [11]).

We have a polynomial $p : \mathbb{C} \to \mathbb{C}$ and a family of functions

$$N_{\mathbf{a}} : \mathbb{C} \times (0, \infty) \to \mathbb{C},$$

$\mathbf{a} \in \mathbb{C}^4 \sim \{\mathbf{0}\}$, whose restriction $N_{\mathbf{a}}(\cdot, L)$ is entire for each $L > 0$. We consider a family of functions $f_{\mathbf{a}}(\cdot, L)$ defined by

$$N_{\mathbf{a}}(\xi, L) = f_{\mathbf{a}}(\xi, L)\, p(\xi) \tag{2}$$

in its maximal domain. For a given polynomial $p(\cdot)$, the problem of characterizing the set of values $L_0 > 0$, for which it is possible to find a non null $\mathbf{a}_0 \in \mathbb{C}^4$ such that the function $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire, is a challenging problem and of particular interest of the academic community.

Vasconcellos and Silva [11, Lemma 2.1] discussed the existence of non-zero solutions for (1) whose energy is constant over time. Their results show that the existence of such solutions is equivalent to determining the lengths of interval $(0, L_0)$ for which it is possible to verify that the condition

$$(\exists\, \lambda \in \mathbb{C},\ u_0 \in (H_0^3(0, L_0) \cap H^5(0, L_0), \mathbb{C}) \ \Rightarrow\ \lambda u_0 + \beta u_0' + \kappa u_0''' + \eta u_0''''' = 0) \tag{3}$$

is valid. Such condition in turn reduces to the problem of characterizing the set $\mathcal{X}$ of $L_0 > 0$ values, for which exist r and $\mathbf{a}_0$ providing that function $f_{\mathbf{a}}(\cdot, L)$ is entire for $L = L_0$ and $\mathbf{a} = \mathbf{a}_0$. In this case, using (2), $f_{\mathbf{a}}(\cdot, L)$ is defined by

$$\begin{aligned} N_{\mathbf{a}}(\xi, L) &= a_1 i\xi - a_2 i\xi e^{-i\xi L} + a_3 - a_4 e^{-i\xi L} \\ p(\xi) &= r + \beta\xi - \kappa\xi^3 + \eta\xi^5 \end{aligned} \tag{4}$$

where $\mathbf{a} = (a_1, a_2, a_3, a_4)$ and $r \in \mathbb{R}$. It follows from (3) that λ is a pure imaginary number. Thus, we only have to consider polynomials $p(\cdot)$ with $r \in \mathbb{R}$.

For each $r \in \mathbb{R}$ and $\mathbf{a}_0 \in \mathbb{C}^4 \sim \{\mathbf{0}\}$, let $\mathcal{X}_{\mathbf{a}_0 r}$ be the set of $L_0 > 0$ values, for which the function $f_{\mathbf{a}}(\cdot, L)$ is entire for $L = L_0$ and $\mathbf{a} = \mathbf{a}_0$. The set $\mathcal{X}$ is the union of $\mathcal{X}_{\mathbf{a}_0 r}$ for $r \in \mathbb{R}$ and $\mathbf{a}_0 \in \mathbb{C}^4 \sim \{\mathbf{0}\}$. Here, we place emphasis on the following statements:

(S1) $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire;

(S2) all the zeros, taking the respective multiplicities into account, of the polynomial p are zeros of $N_{\mathbf{a}_0}(\cdot, L)$;

(S3) the maximal domain of $f_{\mathbf{a}_0}(\cdot, L_0)$ is $\mathbb{C}$;

which are, clearly, equivalent and will be widely used throughout this article. A closer look shows that determining the solution to the problem guarantees the existence of a Möbius transformation in some circumstances. Further, for the function $f_{\mathbf{a}}(\cdot, L)$, defined by (2) and (4), to be entire, given the equivalence between statements (S1) and (S2), informally, we must have

$$\frac{a_1 i\xi_0 + a_3}{a_2 i\xi_0 + a_4} = e^{-iL\xi_0} \tag{5}$$

for each root ξ_0 of the polynomial p. We note that for $\mathbf{a}$ such that $a_1a_3 - a_2a_4 \neq 0$, the left side of (5) suggests that a Möbius transformation is defined. Note that we already have an indication that for a polynomial p with at least two roots differing by an integer multiple of $2\pi/L$, we obtain $L \notin \mathcal{X}$. With this, a method for solving the problem is revealed: we must verify for which structures of the roots of the polynomial p is it possible to define a Möbius transformation M that satisfies $M(\xi_0) = e^{-iL\xi_0}$ for each zero ξ_0 of polynomial p.

Taking (5), it is essential to define, for each non null $\mathbf{a} \in \mathbb{C}^4$, the discriminant of $\mathbf{a}$, specifically, the complex number $d(\mathbf{a}) = a_1a_4 - a_2a_3$. It is natural, however, to consider: (i) $d(\mathbf{a}) = 0$ or (ii) $d(\mathbf{a}) \neq 0$.

The main result shown in this article guarantees that the existence of pairs $(\mathbf{a}_0, L_0)$ that make $f_{\mathbf{a}}(\cdot, L)$ entire is intimately linked to whether or not the discriminant is zero. In fact, when the discriminant of $\mathbf{a}$ is zero, such pairs do not exist for any $r \in \mathbb{R}$. On the other hand, if the discriminant of $\mathbf{a}$ is non-zero, we identify situations where the pairs $(\mathbf{a}_0, L_0)$ can exist or not. Whereas case (i) has been completely solved here, in case (ii) there are situations where the problem remains to be solved, i.e., in some cases, we do not know whether or not it is possible to satisfy (5). As far as we know, Rosier [9] was the first to analyze these kinds of problems. In fact, he showed that the existence of non-trivial solutions for the Kortweg de Vries equation, whose energies do not decay over time, is equivalent to determining the set $\mathcal{U}$ of values $l_0 > 0$, for which there exists a non null $\mathbf{k}_0 \in \mathbb{C}^2$ and $s \in \mathbb{C}$, so that the function $g_{\mathbf{k}}(\cdot, L)$ with $\mathbf{k} = (k_1, k_2)$, defined by

$$M_{\mathbf{k}}(\xi, l) = g_{\mathbf{k}}(\xi, l)\, q(\xi), \tag{6}$$

is entire for $\mathbf{k} = \mathbf{k}_0$ and $l = l_0$. Here, in particular, $M_{\mathbf{k}}(\xi, l) = k_1 - k_2 e^{-iL\xi}$ and $q(\xi) = \xi^3 - \xi + s$. Then Rosier [9] proves that

$$\mathcal{U} = \Big\{ 2\pi\sqrt{\frac{m^2 + mn + n^2}{3}} \, : \; n, m \in \mathbb{N} \Big\}.$$

Let us take case (i) from the same starting point as Rosier [9], i.e., the analysis of zeros of $N_{\mathbf{a}}(\cdot, L)$. Here, it makes no sense to argue about the existence of a Möbius transformation. Case (ii) is completely based on equation (5). Our strategy is quite efficient. It proved to be efficient in this situation, where using previously established results, such as the Weierstrass factorization theorem, is not possible.

Notice that, for each choice of the coefficients β, κ and η, condition (3) associates the Kawahara equation $u_t + \beta u_x + \kappa u_{xxx} + \eta u_{xxxxx} = 0$ to a family of polynomials $p(\xi) = r + \beta\xi - \kappa\xi^3 + \eta\xi^5, \quad r \in \mathbb{R}$.

Let $\mathcal{X}$ be the set of the lengths of interval $(0, L_0)$ for which exist non-zero solutions for (1) whose energy is constant over time. Consider for each $r \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{C}^4 \sim \{\mathbf{0}\}$, the set $\mathcal{X}_{\mathbf{a}r}$ of values $L_0 > 0$ for which the function $f_{\mathbf{a}}(\cdot, L)$ defined by (2) and (4) is entire for $L = L_0$. We can decompose $\mathcal{X}$ as the union of the sets $\mathcal{X}_{\mathbf{a}r}$ for $r \in \mathbb{R}$ and non null $\mathbf{a} \in \mathbb{C}^4$

We extend the results obtained by Vasconcellos and Silva [11, 12] for characterizing the set $\mathcal{X}$ for the Kawahara equation (1). They have partially analyzed the case $\kappa = 0$ in (1) and did not deal with the case $\kappa = 1$ in (1). In our proof, we argue by exhaustion characterizing the sets $\mathcal{X}_{\mathbf{a}r}$. In Part (I) of Theorem 1.1, we see that if $d(\mathbf{a}) = 0$, then $\mathcal{X}_{\mathbf{a}r} = \emptyset$ for all $r \in \mathbb{R}$. As a consequence of this result, it follows that for any Kawahara equation (1), the set $\mathcal{X}$ is given by the union of the set $\mathcal{X}_{\mathbf{a}r}$ for $r \in \mathbb{R}$ and $d(\mathbf{a}) \neq 0$. Our results for $d(\mathbf{a}) \neq 0$ allow to partially describe the set $\mathcal{X}$ for Kawahara equations (1) with $\beta = 1$ and $\kappa \neq 0$ or $\beta = 0$ and $\kappa < 0$. For Kawahara equations (1) with $\beta = 0$, $\kappa > 0$, as a consequence of Theorem 1.1, we obtain that $\mathcal{X}$ is empty.

Now we summarize the results obtained in this article in the following theorem guided by the roots of polynomial p, as we will shortly see.

Theorem 1.1. *Let $r \in \mathbb{R}$, a non null $\mathbf{a} \in \mathbb{C}^4$ and $L > 0$, and consider the function $f_{\mathbf{a}}(\cdot, L)$ defined by the product*

$$N_{\mathbf{a}}(\xi, L) = f_{\mathbf{a}}(\xi, L)\, p(\xi) \tag{7}$$

in its maximal domain. Let us suppose that $N_{\mathbf{a}}(\xi, L)$ and $p(\xi)$ are as in (4). Let $\mathcal{X}_{\mathbf{a}r}$ be the set of values $L_0 > 0$ for which the function $f_{\mathbf{a}}(\cdot, L)$ defined by (7) is entire for $L = L_0$.

(I) If $L_0 > 0$ is such that $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire for some non null $\mathbf{a}_0 \in \mathbb{C}^4$, then $d(\mathbf{a}_0) \neq 0$. In other words, for any non null $\mathbf{a}$, if $d(\mathbf{a}) = 0$, we obtain $\mathcal{X}_{\mathbf{a}r} = \emptyset$, for any $r \in \mathbb{R}$. The reciprocal, however, is false.

(II) If $\mathbf{a}$ is such that $d(\mathbf{a}) \neq 0$ and one of following three items occurs:

- *(a) $\beta = 1$ and $|r| > z - \kappa z^3 + \eta z^5$, where $z = \sqrt{\frac{3\kappa - \sqrt{9\kappa^2 - 20\eta}}{10\eta}}$;*
- *(b) $\beta = 0$, $\kappa > 0$ and $r \in \mathbb{R}$;*
- *(c) $\beta = 0$, $\kappa < 0$ and $|r| > -\kappa z^3 + \eta z^5$, where $z = \sqrt{\frac{3\kappa}{5\eta}}$.*

Then there is no $L > 0$ that renders the function $f_{\mathbf{a}}(\cdot, L)$ entire. Therefore, $\mathcal{X}_{\mathbf{a}r} = \emptyset$.

(III)

- *(a) If $\beta = 1$ and $r = 0$, then there exist $L_0 > 0$ and non null $\mathbf{a}_0$ such that $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire if and only if*

$$L_0 \in \left\{ L \in \mathbb{R},\ k \operatorname{cotanh}\left(\frac{Lk}{2}\right) = -\rho \cot\left(\frac{L\rho}{2}\right) \right\}$$

where

$$\rho = \sqrt{\frac{\kappa - \sqrt{\kappa^2 - 4\eta}}{2\eta}} \quad \text{and} \quad k = \sqrt{\left|\frac{\kappa + \sqrt{\kappa^2 - 4\eta}}{2\eta}\right|}.$$

- *(b) If $\beta = 0$, $\kappa < 0$ and $r = 0$, then there exist $L_0 > 0$ and non null $\mathbf{a}_0$ such that $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire if and only if*

$$L_0 \in \left\{ L > 0,\ \tan\frac{\rho L}{2} = \frac{\rho L}{2} \right\},$$

where $\rho = \sqrt{\kappa/\eta}$.

The sets in (a) and (b) are enumerable.

The knowledge of the zeros of $g_{\mathbf{k}}(\xi, l)$ in (6) plays a key role in the Rosier's analysis of the existence of non-trivial solutions for the Kortweg de Vries equation, whose energies do not decay over time. The function $f_{\mathbf{a}}(\xi, L)$ related to Kawahara equation does not resemble this fact and its structure together with the order of the polynomial turn the analysis of the Kawahara case into a hard problem. Many other authors have made efforts to tackle this problem (see for instance, Glass and Guerrero [5], for a particular case of (III)(b); Araruna, Capistrano-Filho and Doronin [1], for an example of a critical set). Our results take their contributions into account. We show they can be presented and obtained in a systematic way and we go a step further.

2 Auxiliary results

We establish some lemmas needed for proving the three parts of Theorem 1.1.

Part (I)

The main idea behind Part (I) of Theorem 1.1 is to find out whether there is at least one zero of polynomial p that is not a zero of $N_{\mathbf{a}}(\cdot, L)$. The following lemma is a decisive factor in obtaining this result.

Lemma 2.1. *Let non null $\mathbf{a} \in \mathbb{C}^4$ with $d(\mathbf{a}) = 0$ and $L > 0$. Then the set of the imaginary parts of the zeros of $N_{\mathbf{a}}(\cdot, L)$ has at most two elements.*

Part (II)

The following lemma essentially states that if the polynomial p has "too many" complex roots, equation (5) cannot be satisfied.

Lemma 2.2. *For any $L > 0$, there is no Möbius transformation M such that*

$$M(\xi) = e^{-iL\xi}, \quad \xi \in \{\xi_1, \xi_2, \overline{\xi_1}, \overline{\xi_2}\}$$

with $\xi_1, \xi_2, \overline{\xi_1}, \overline{\xi_2}$ all distinct in $\mathbb{C}$.

Part (III)

Lemma 2.3 below, unlike Lemma 2.2, guarantees the existence of a Möbius transformation in a case when the polynomial p has exactly three real roots whose multiplicities are equal to 1. Lemma 2.5 below guarantees the existence of a Möbius transformation when all roots of polynomial p are real.

Lemma 2.3. *Let L, k and ρ be real numbers with $L > 0$ and $k \neq 0$. There is a unique Möbius transformation M which satisfies $M(0) = 1$, $M(\pm\rho) = e^{\mp iL\rho}$ and $M(\pm ik) = e^{\pm Lk}$ if and only if the following equality occurs*

$$k \operatorname{cotanh}\left(\frac{Lk}{2}\right) = -\rho \cot\left(\frac{L\rho}{2}\right).$$

Lemma 2.4. *Let k and ρ be real numbers with $k \neq 0$. The positive solutions of the equation*

$$k \operatorname{cotanh}\left(\frac{Lk}{2}\right) = -\rho \cot\left(\frac{L\rho}{2}\right)$$

form a countable set.

Lemma 2.5. *Consider the family of functions $f_{\mathbf{a}}(\cdot, L)$ defined by (4) and let us suppose that $r = \beta = 0$. Let $\mathcal{X}_{\mathbf{a}r}$ be the set of values $L_0 > 0$ for which the function $f_{\mathbf{a}}(\cdot, L)$ is entire for $L = L_0$.*

1. *If $\kappa < 0$, there exist $L_0 > 0$ and non null $\mathbf{a}_0$ such that $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire if and only if*

$$L_0 \in \left\{L > 0, \tan\frac{\rho L}{2} = \frac{\rho L}{2}\right\}$$

   where $\rho = \sqrt{\kappa/\eta}$.

2. *If $\kappa > 0$, then $f_{\mathbf{a}}(\cdot, L)$ is not entire for any non null $\mathbf{a}$ and $L > 0$. That is, $\mathcal{X}_{\mathbf{a}0} = \emptyset$.*

2.1 Describing the roots

We also need a lemma that separates the roots of the polynomial $p(\xi) = r + \beta\xi - \kappa\xi^3 + \eta\xi^5$ (where r, β, κ and η are real such that $\beta \in \{0, 1\}$, $\kappa \neq 0$ and $\eta < 0$) into groups according to their algebraic structure. To characterize a group, we consider the quantity of real roots and their respective multiplicities. It is worth noting that for each polynomial p, the relation of its roots with these groups determines whether or not a solution exists for the problem of determining an entire member of family $f_{\mathbf{a}}(\cdot, L)$.

3 Final Remarks

It is worth noting that when $d(\mathbf{a}) = 0$, the sets $\mathcal{X}_{\mathbf{a}r}$ were completely characterized for any $r \in \mathbb{R}$. The same happens when we consider $d(\mathbf{a}) \neq 0$, $\beta = 0$, $\kappa > 0$ and $r \in \mathbb{R}$; i.e., when $p(\xi) = r - \kappa\xi^3 + \eta\xi^5$. In particular, in this case, Theorem 1.1 tells us that the sets $\mathcal{X}_{\mathbf{a}r}$ are empty for all non null $\mathbf{a} \in \mathbb{C}^4$ and $r \in \mathbb{R}$. Thus the set $\mathcal{X}$ is empty and the problem of the initial and boundary value, analyzed by Vasconcellos and Silva [11] and associated with the linear Kawahara equation $u_t + \kappa u_{xxx} + \eta u_{xxxxx} = 0$, does not admit non-trivial solutions whose energies do not decay over time. Note that for $p(\xi) = r + \beta\xi - \kappa\xi^3 + \eta\xi^5$, the case $d(\mathbf{a}) \neq 0$ remains to be solved in two situations: (a) when $r \neq 0$ and p has exactly three real roots, with all the multiplicities being equal to 1 and (b) p has exactly three real roots with one of them having a multiplicity of 2.

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