Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 5, N. 1, 2017.

Trabalho apresentado no CNMAC, Gramado - RS, 2016.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

A Note on Self-Similar Solutions of the Curve Shortening Flow

Márcio R. Adames[1]
Departamento Acadêmico de Matemática, UTFPR, Curitiba, PR

Abstract. This article gives an alternative approach to the self-shrinking and self-expanding solutions of the curve shortening flow, which are related to singularity formation of the mean curvature flow. Further we describe the self-similar solutions in terms of a simple ODE and give an alternative proof that they lie in planes.

Keywords. Curve Shortening Flow, Abresch & Langer Curves, Planar Solutions

1 Introduction

To deform a curve (usually smooth) by the curve shortening flow (CSF) is to let it evolve in the direction of its curvature vector, thus generating a family of curves. The problem of understanding the behavior of such family was first addressed by Mullins [9] in 1956 to study ideal grain boundary motion in two dimensions. Renewed interest in the topic came with the works of Gage and Hamilton (e. g. [4], where they show that convex plane curves shrink to a point, becoming more circular as time advances) and Grayson (e. g. [5]). Since then the problem has been studied by many, and of particular significance has been the study of singularity formation.

The present work does not purport to contain a comprehensive introduction to the CSF because of the great number of contributions to the subject (e. g. “The curve shortening flow” of Chou and Zhu [3] contains 113 items in its bibliography). As an important result we cite the complete classification of closed plane curves which shrink under the CSF by Abresch and Langer [1].

This work was somewhat inspired by the recent works of Halldorsson [6], which classifies self-similar (in a broader context) plane curves of the CSF, and Altschuler, Altschuler, Angenent and Wu [2], which provides a classification of self-similar solutions (or solitons) of the CSF in $\mathbb{R}^n$. Both works are based in ODE techniques and the last of them mentions the well known fact, that dilating solitons are planar. In sections 4 and 5 we prove that shrinking and expanding solitons are planar.

[1]marcioadames@utfpr.edu.br

2 Plane self-shrinkers.

Definition 2.1. A family $\gamma : (a, b) \times I \to \mathbb{R}^n$ of smooth immersions $\gamma_t : I \to \mathbb{R}^n$, evolves by the *curve shortening flow* (CSF) if it satisfies

$$\left(\frac{\partial \gamma}{\partial t}\right)^{\perp} = \frac{\partial^2 \gamma}{\partial s^2}, \tag{1}$$

where s is the arc length parameter (not necessarily the parameter of I) of the curve γ_t.

Given an initial curve $\gamma_0 : I \to \mathbb{R}^n$, if there is a unique family $\gamma : [0, \epsilon) \times I \to \mathbb{R}^n$ satisfying $\gamma(0, \cdot) = \gamma_0$ (to find such a family is to locally solve a P.D.E), we say that the curve shortening flow is deforming the initial curve.

The present work discusses a special class of curves that is deformed by the curve shortening flow only by changing its size, and not its shape. These curves are said self-similar solutions to the CSF.

Let $\gamma : I \to \mathbb{R}^2$ be a self-similar shrinking solution of the curve shortening flow that is parametrized by arc length. Thus

$$\gamma'' = -\gamma^{\perp} = \langle \gamma, \gamma' \rangle \gamma' - \gamma. \tag{2}$$

Lemma 2.1. *The only self-shrinkers (solution of eq.* (2)*) that pass through the origin are the straight lines.*

Proof. If $\gamma(t_0) = 0$ and $\gamma'(t_0) = \overrightarrow{v}$, then $\|\overrightarrow{v}\| = 1$, for the curve is parametrized by arc length. It follows that $\beta(t) = (t - t_0)\overrightarrow{v}$ satisfies

$$\beta''(t) = 0,$$

and

$$\langle \beta, \beta' \rangle \beta' - \beta = (t - t_0)\overrightarrow{v} - (t - t_0)\overrightarrow{v} = 0.$$

Therefore $\beta(t)$, $t \in \mathbb{R}$, is a solution to (2) with $\beta(t_0) = \gamma(t_0) = 0$ and $\beta'(t_0) = \gamma'(t_0) = \overrightarrow{v}$. From the uniqueness of the solutions to the associated (with eq. (2)) initial value problem it follows that $\gamma(t) = \beta(t)$. □

The straight lines are static under the curve shortening flow. As the other solutions do not cross the origin, we can write them in polar coordinates. We follow calculating

$$\langle \gamma, \gamma \rangle'' = 2\langle \gamma'', \gamma \rangle + 2\langle \gamma', \gamma' \rangle,$$

and, writing $\alpha = \langle \gamma, \gamma \rangle$, we get in view of eq. (2)

$$\alpha'' - \frac{(\alpha')^2}{2} + 2\alpha = 2. \tag{3}$$

The associated initial value problem admits an unique solution. Further there are solutions of eq. (3) that are always positive:

Lemma 2.2. *A solution of eq.* (3) *with* $0 < \alpha(0) < 1$ *and* $\alpha'(0) = 0$ *is strictly positive.*

Proof. First note that $\alpha(t)$ has a local minimum at $t = 0$, so that if there is $t_1 \in D(\alpha)$ such that $\alpha(t_1) \leq \alpha(0)$, then there would be a local maximum at some $t_0 \in (0, t_1)$. But

$$\beta(t) := \alpha(t_0 - t),$$

also satisfies eq. (3) and $\beta(0) = \alpha(t_0)$, $\beta'(0) = \alpha'(t_0)$ and $\beta''(0) = \alpha''(t_0)$. Thus a solution of eq. (3) would exist for all $t \in \mathbb{R}$ and be given by

$$\alpha^*(t) = \begin{cases} \alpha(t - 2nt_0), & t \in [2nt_0, (2n+1)t_0) \\ \beta(t - 2nt_0) = \alpha((2n+1)t_0 - t), & t \in [(2n+1)t_0, (2n+2)t_0) \end{cases}$$

so that min $\alpha(t) = \alpha(0)$. □

The figure below illustrates the construction of a solution to eq. (3). Further, the periodicity of the solution is expected from the actual form of the Abresch & Langer curves.

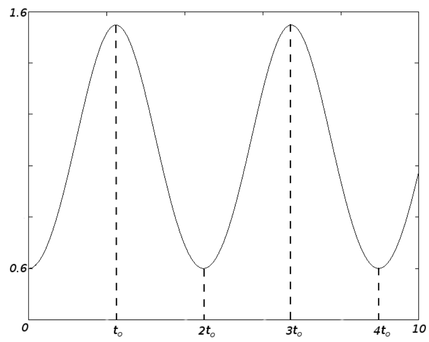

Figure 1: A solution $\alpha(t)$, with $\alpha(0) = 0.6$ and $\alpha'(0) = 0$

For every solution of eq. (3) that is positive, it is possible to define a function $u = \sqrt{\alpha}$ and write the self-shrinker in polar coordinates:

$$\gamma(t) = u(t)(\cos(\theta(t)), \sin(\theta(t))). \tag{4}$$

Beyond this $u = \sqrt{\alpha}$ implies

$$\alpha' = 2uu' \qquad \text{and} \qquad \alpha'' = 2u''u + 2(u')^2$$

so that equation (3) turns into

$$u''u + (u')^2 - [u']^2u^2 + u^2 = 1. \tag{5}$$

Further it holds

$$\gamma' = u'(\cos\theta, \sin\theta) + u\theta'(-\sin\theta, \cos\theta) \tag{6}$$

$$\gamma'' = [u'' - u[\theta']^2](\cos\theta, \sin\theta) + [2u'\theta' + u\theta''](-\sin\theta, \cos\theta) \tag{7}$$

$$-\gamma^\perp = [u[u']^2 - u](\cos\theta, \sin\theta) + [u^2u'\theta'](-\sin\theta, \cos\theta), \tag{8}$$

DOI: 10.5540/03.2017.005.01.0022

Therefore equation (2) holds if, and only if, both equations hold:

$$u'' - u[\theta']^2 = u[u']^2 - u, \tag{9}$$
$$2u'\theta' + u\theta'' = u^2 u'\theta' \tag{10}$$

where u is a known function and, recalling that $\|\gamma'\| = 1$,

$$[\theta']^2 = \frac{1 - [u']^2}{u^2} \tag{11}$$

and

$$\theta = \int \frac{4\alpha(t) - (\alpha'(t))^2}{4\alpha^2(t)} dt. \tag{12}$$

In figure 2 there are plots of self-shrinkers constructed from numerical solutions of eqs. (3) and (12). It is not clear which initial conditions generate closed curves.

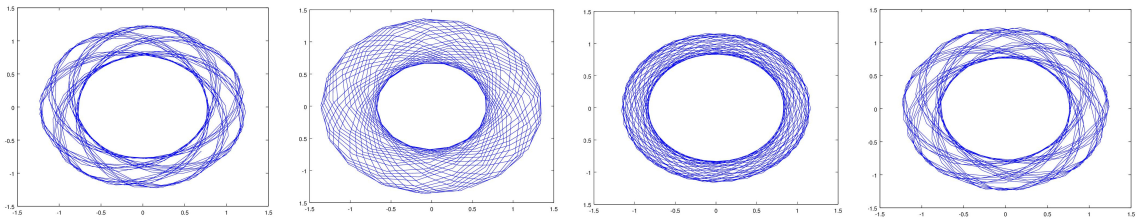

Figure 2: Noncompact Abresch & Langer Curves.

It is not hard to see that any solutions u and θ of equations (5) and (11) also satisfy eq. (9) and (10) and thus generate self-shrinkers of the curve shortening flow through equation (4):

Theorem 2.1. *A curve $\mathcal{C}$ parametrized by $\gamma : I \to \mathbb{R}^2$, $\gamma(t) = \sqrt{\alpha(t)}(cos(\theta(t)), sin(\theta(t)))$ is a self-shrinker of the curve shortening flow if, and only if,*

1. *it is a straight line or*
2. *$\alpha(t) > 0$ for all $t \in I$ and*

$$\alpha'' - \frac{(\alpha')^2}{2} + 2\alpha = 2,$$
$$\theta = \int \frac{4\alpha(t) - (\alpha'(t))^2}{4\alpha^2(t)} dt.$$

3 Plane self-shrinkers.

Consider now a self-similar solution of the curve shortening flow $\gamma : I \to \mathbb{R}^3$ that is parametrized by arc length, then $\alpha = \langle \gamma, \gamma \rangle$ also satisfies eq. (3). Denoting $u = \sqrt{\alpha}$

and taking a positive solution O.D.E. (3) one can write the self-shrinker in spherical coordinates:

$$\gamma(t) = u(\cos\theta(t)\sin\varphi(t), \sin\theta(t)\sin\varphi(t), \cos\varphi(t)).$$

We use the following moving frame to calculate γ'' and $\gamma^{\perp}$:

$$X = \begin{pmatrix} \cos\theta\sin\varphi \\ \sin\theta\sin\varphi \\ \cos\varphi \end{pmatrix}, \qquad \frac{\partial X}{\partial\theta} = \begin{pmatrix} -\sin\theta\sin\varphi \\ \cos\theta\sin\varphi \\ 0 \end{pmatrix}, \qquad \frac{\partial X}{\partial\varphi} = \begin{pmatrix} -\cos\theta\cos\varphi \\ -\sin\theta\cos\varphi \\ -\sin\varphi \end{pmatrix}.$$

Then:

$$\gamma'(t) = u'X + u\theta'\frac{\partial X}{\partial\theta} + u\varphi'\frac{\partial X}{\partial\varphi},$$

$$\begin{aligned}\gamma''(t) = &\left[u'' - u[\theta']^2\sin^2\varphi - u[\varphi']^2\right]X + \left[2u'\varphi' - u[\theta']^2\sin\varphi\cos\varphi + u\varphi''\right]\frac{\partial X}{\partial\varphi} \\ &+ \left[2u'\theta' + u\theta'' + u\theta'\varphi'\frac{\cos\varphi}{\sin\varphi} + u\varphi'\theta'\frac{\cos\varphi}{\sin\varphi}\right]\frac{\partial X}{\partial\theta}\end{aligned}$$

and

$$\gamma^{\perp} = uX - uu'\left[u'X + u\theta'\frac{\partial X}{\partial\theta} + u\varphi'\frac{\partial X}{\partial\varphi}\right].$$

In this fashion eq. (2) implies that

$$\begin{aligned}&u'' - \sin^2\varphi u[\theta']^2 - u[\varphi']^2 = -u + u[u']^2, \\ &2u'\theta' + u\theta'' + u\theta'\varphi'\frac{\cos\varphi}{\sin\varphi} + u\varphi'\theta'\frac{\cos\varphi}{\sin\varphi} = u^2u'\theta', \\ &2u'\varphi' - u[\theta']^2\sin\varphi\cos\varphi + u\varphi'' = u^2u'\varphi'\end{aligned}$$

and, as we chose a parametrization by arc length,

$$[u']^2 + [u\theta']^2\sin^2\varphi + [u\varphi']^2 = 1.$$

Numerical evaluation of these equations indicate that all self-shrinkers in $\mathbb{R}^3$ lie in planes:

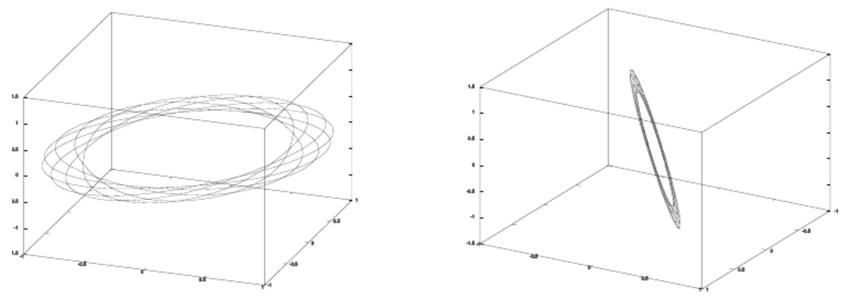

Figure 3: Two plots of the same self-shrinker from different angles.

4 Self-shrinking curves in $\mathbb{R}^n$

In this section we prove:

Theorem 4.1. *Every self-shrinking solution of the curve shortening flow $\gamma : I \to \mathbb{R}^n$ lies in a plane.*

Proof. First of all let γ be parametrized by arc length. Then, by eq. (2),

$$\begin{aligned}\gamma''' =& \gamma' - \gamma' + \langle \gamma, \gamma'' \rangle \gamma' + \langle \gamma, \gamma' \rangle \gamma'' \\ =& - \langle \gamma, \gamma \rangle \gamma' + \langle \gamma, \gamma' \rangle^2 \gamma' + \langle \gamma, \gamma' \rangle \gamma'' \\ =& - \|\gamma''\|^2 \gamma' + \langle \gamma, \gamma' \rangle \gamma''.\end{aligned}$$

If $r, s : (a, b) \to \mathbb{R}$ are solutions to

$$(r\gamma' + s\gamma'')' = 0, \tag{13}$$

then the vector field $v(t) = r(t)\gamma'(t) + s(t)\gamma''(t)$ over $\gamma(a, b)$ is a constant vector. Note that eq. (13) implies

$$r'\gamma' + s'\gamma'' + r\gamma'' + s(-\|\gamma''\|^2\gamma' + \langle \gamma, \gamma' \rangle \gamma'') = 0.$$

So that, if $\gamma' \neq 0$ and $\gamma'' \neq 0$, r and s satisfy the following O.D.E system:

$$\begin{cases} r'(t) = s(t)(\langle \gamma, \gamma \rangle - \langle \gamma, \gamma' \rangle^2), \\ s'(t) = -s(t)\langle \gamma, \gamma' \rangle - r(t). \end{cases} \tag{14}$$

The associated initial value problem has a unique solution for every fixed pair of values for $r(t_0)$ and $s(t_0)$, which can be extended for the whole domain of γ, and any solution to eq. (14) makes eq. (13) hold. Thus $r\gamma' + s\gamma''$ is a constant vector. Further, if the curve defined by γ is not a straight line or is degenerate to a point, then there is $t_0 \in (a, b)$ such that $\gamma'(t_0) \neq 0$ and $\gamma''(t_0) \neq 0$. Letting $r(t_0)$ and $s(t_0)$ vary makes $v(t_0) = r(t_0)\gamma'(t_0) + s(t_0)\gamma''(t_0)$ equal to any vector in the plane defined by $\gamma'(t_0)$, $\gamma''(t_0)$ and the origin.

Furthermore $v(t) = r(t)\gamma'(t) + s(t)\gamma''(t) = r(t_0)\gamma'(t_0) + s(t_0)\gamma''(t_0) = v(t_0)$ for all $t \in (a, b)$. Thence the family of $v(t)$ thus obtained spans the same plane for any t. There are linearly independent vectors in this family, so that $\gamma'(t)$ can be written as a linear combination of two vectors of the like, then $\gamma'(t)$ is always on this plane and curve lies in a plane. □

5 Self-expanders

Let $\gamma : I \to \mathbb{R}^2$ be a self-similar expanding solution of the curve shortening flow that is parametrized by arc length. Then

$$\gamma'' = \gamma^{\perp} = \gamma - \langle \gamma, \gamma' \rangle \gamma' \tag{15}$$

In analogous fashion to the self-shrinking curves one can find:

Theorem 5.1. *A curve $\mathcal{C}$ parametrized by $\gamma : I \to \mathbb{R}^2$, $\gamma(t) = \sqrt{\alpha(t)}(cos(\theta(t)), sin(\theta(t)))$ is a self-expander of the curve shortening flow if, and only if,*

1. *it is a straight line or*
2. *$\alpha(t) > 0$ for all $t \in I$ and*

$$\alpha'' + \frac{(\alpha')^2}{2} - 2\alpha = 2,$$
$$[\theta']^2 = \frac{1 - [u']^2}{u^2}.$$

Furthermore, calculations analogous to the previous sections, show that the self-expanders are also necessarily planar:

Theorem 5.2. *Every self-expanding solution of the curve shortening flow $\gamma : I \to \mathbb{R}^n$ lies in a plane.*

Acknowledgments

The author thanks André Fabiano Steklain Lisboa and Adara Testa Gomes for suggestions to the text.

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