Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 5, N. 1, 2017.

Trabalho apresentado no CNMAC, Gramado - RS, 2016.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Solvability of a homogeneous third-order evolution equation by inverse scattering

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The authors in [5], while developing a genetic program to re-obtain differential equations from known solutions, tested a very famous solitary wave of the KdV equation and surprisingly obtained a different equation. After introducing two arbitrary parameters a and ϵ, they obtained the equation

$$u_t + 2a\frac{u_x u_{xx}}{u} - \epsilon a u_{xxx} = 0. \tag{1}$$

The choice $\epsilon a = 1$, $a = 3/2$ makes equation (1) relate to KdV and mKdV equations through Miura-type transformations, see [1]. One of the main features of both KdV and mKdV equations is the existence of pairs of pseudo-differential operators such that the equations can be written as a compatibility condition of two linear equations. The existence of these pairs leads to the existence of an infinite number of conserved densities, a very important physical property that has been the subject of intense research.

We say an equation is solvable by the inverse scattering method if there exist two differential operators $\mathcal{L}$ and $\mathcal{B}$ such that the system $\mathcal{L}\varphi = \lambda\varphi,\ \varphi_t = \mathcal{B}\varphi$ is solved in the solutions of the equation, for all spectral parameters λ, through the Lax equation $\mathcal{L}_t = [\mathcal{B}, \mathcal{L}]$. The operators $\mathcal{L}, \mathcal{B}$ are then said to form a Lax pair for the equation under consideration.

The existence of such pair guarantees, see [2], the existence of a recursion operator, which always maps a symmetry into another symmetry. However, we observed that equation (1) will only admit recursion operators if $\epsilon a = 1$ and $a = \pm 3/2$, see [1, 3, 4]. Therefore, it is natural to only look for Lax pairs of equation (1) for $a = \pm 3/2$ and $\epsilon a = 1$.

For that purpose, consider the differential operators

$$\mathcal{L} = AD_x^2 + C[u], \quad \mathcal{B} = \alpha D_x^3 + \gamma[u]D_x + \frac{1}{2}D_x\gamma[u],$$

where A, α are constants, $u = u(x,t)$ and $C[u], \gamma[u]$ are functions depending on u and its derivatives. As $\mathcal{L}$ is a second-order differential operator, defining $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \varphi_x \\ \varphi_{xx} \end{bmatrix}$,

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the system can be rewritten as the matrix representation $\phi_x = U\phi,\ \phi_t = V\phi$, where

$$U = \begin{bmatrix} 0 & 1 \\ \frac{1}{A}(\lambda - c) & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} \frac{1}{2}\gamma_x - \frac{\alpha}{A}C_x & \gamma + \frac{\alpha}{A}(\lambda - C) \\ \frac{1}{2}\gamma_{xx} - \frac{\alpha}{A}C_{xx} + \frac{1}{A}(\lambda - C)\gamma + \frac{\alpha}{A^2}(\lambda - C)^2 & \frac{3}{2}\gamma_x - \frac{2\alpha}{A}C_x \end{bmatrix}.$$

Lax equation is transformed into a zero-curvature representation $\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0$, which reads

$$\begin{bmatrix} 0 & 0 \\ -\frac{C_t}{A} + \frac{\alpha}{4A}C_{xxx} + \frac{3\alpha}{2A^2}CC_x & 0 \end{bmatrix} = 0. \tag{2}$$

The difficulty of obtaining a Lax pair is having the änsatz of which function C to choose. In the particular case $a = 3/2$, the choices $C = u_{xx}/u$, $A = -1$ and $\alpha = 4$ show that

$$-\frac{C_t}{A} + \frac{\alpha}{4A}C_{xxx} + \frac{3\alpha}{2A^2}CC_x = \left(\frac{1}{u}D_x^2 - \frac{u_{xx}}{u^2}\right)\left(u_t + 3\frac{u_x u_{xx}}{u} - u_{xxx}\right),$$

which shows that equation (1) with $\epsilon a = 1$ and $a = 3/2$ is solvable by the inverse scattering method. The case $a = -3/2$ is a little more difficult and has not been solved yet. Moreover, it has not been clear yet if it will admit a Lax pair at all.

Agradecimentos

P. L. da Silva would like to thank CAPES for her scholarship. I. L. Freire thanks FAPESP (2014/05024-8) and CNPq (308941/2013-6) for financial support.

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